Queen's University - Math 844 **Fall 2022**

Problem Set #6

Posted: Friday, 2/12/2022

Due: Friday, 9/12/2022

1. Let M be a C^∞ $n-$manifold; show that the tangent bundle TM of M is orientable. (Hint: Consider the C^∞ atlas on TM derived from a C^∞ atlas on M).
2. Let G be a Lie group. Show that G is orientable.
3. Let M be a C^∞ $n-$manifold, and let $\omega \in \Omega^2(M)$. The pair (M, ω) is called a symplectic manifold if:

   (i) ω is closed (i.e. $d\omega = 0$), and

   (ii) ω is non-degenerate; i.e. $\forall p \in M$: $\omega(\mathbf{v}, \mathbf{w}) = 0$, $\forall \mathbf{w} \in T_pM \Rightarrow \mathbf{v} = 0$.

   ω is then called a symplectic structure on M.

   (a) Show that if M admits a symplectic structure, then necessarily M is even-dimensional.

   (b) Show that if M admits a symplectic structure, then necessarily M is orientable.

   (c) Let M be an $n-$dimensional manifold, and consider the cotangent bundle $\pi : T^\star M \to M$. Let $\lambda \in \Omega^1(T^\star M)$ be defined by: $\forall (p, \alpha) \in T^\star M$, $\forall \mathbf{v} \in T_{(p,\alpha)}(T^\star M)$, $\langle \lambda_{(p,\alpha)}, \mathbf{v}\rangle = \langle \alpha_p, \mathbf{T}_{(p,\alpha)}\pi(\mathbf{v})$ (λ is called the Liouville $1-$form on $T^\star M$). Let $\omega = d\lambda$; show that $(T^\star M, \omega)$ is a symplectic manifold. (ω is called the canonical symplectic structure on $T^\star M$).

   (d) Consider $\mathbb{R}^n$ with the canonical global coordinate system $(q^1, \cdots, q^n)$, and let $(q^1, \cdots, q^n, p_1, \cdots, p_n)$ be the corresponding global chart of $T^\star\mathbb{R}^n$ (i.e. the element of $T^\star\mathbb{R}^n$ having coordinate values $(a^1, \cdots, a^n, b_1, \cdots, b_n)$ is precisely the pair $(a, \sum_{i=1}^n b_i dq^i(a))$, where $a = (a^1, \cdots, a^n)$. . Express the canonical symplectic structure of $T^\star\mathbb{R}^n$ using the coordinate functions $(q^1, \cdots, q^n, p_1, \cdots, p_n)$. (Note the choice of p's and q's for the coordinate functions ... historical connections to classical mechanics: The q^i are the "position" variables, the p_i the "momentum" variables.)

   (e) Let (M, ω) be a symplectic manifold, and let $H : M \to \mathbb{R}$ be a C^∞ function (the "Hamiltonian"). We define a vector field X_H on M as follows: $\omega_p(X_H(p), \mathbf{v}) = dH(p)$, $\forall p \in M$, $\forall \mathbf{v} \in T_pM$. Show that X_H is well-defined on M and C^∞ (X_H is called the Hamiltonian vector field corresponding to the Hamiltonian H).

   (f) Consider now $T^\star\mathbb{R}^n$ with its canonical symplectic structure; let $H \in C^\infty(T^\star\mathbb{R}^n)$ be the Hamiltonian, and let X_H denote the corresponding Hamiltonian vector field. Write the equations for the integral curves of X_H in the canonical global chart of $T^\star\mathbb{R}^n$ (Hamilton's equations).
4. Recall that a subset F of $\mathbb{R}^n$ has Lebesgue measure zero if $\forall \epsilon > 0$ there exists a countable cover $(B_i)_{i\in\mathbb{N}}$ of F by open balls B_i, with respective radii r_i, such that $\sum_{i=0}^\infty r_i^n < \epsilon$. Let now M be a C^∞ $n-$manifold, and let $F \subset M$.

   (a) Show that the notion of F having (Lebesgue) measure zero is well-defined.

   (b) Let F have zero measure, and let $\omega \in \Omega_c^n(M)$. Show that $\int_{M\setminus F}\omega = \int_M \omega$.
5. Let $n \in \mathbb{N}^\star$, and consider the inclusion mapping $S^n \hookrightarrow \mathbb{R}^{n+1} \setminus \{0\}$, where S^n is the unit sphere of $\mathbb{R}^{n+1} \setminus \{0\}$. Consider the C^∞ $n-$form on $\mathbb{R}^{n+1} \setminus \{0\}$ defined by $\omega_0 = \sum_{i=0}^n (-1)^i x^i dx^0 \wedge \cdots \wedge \widehat{dx^i} \cdots \wedge dx^n$, and recall that the C^∞ $n-$form on S^n defined by $\omega_{S^n} = i^\star(\omega_0)$ is an orientation form on S^n. Compute $\int_{S^n}\omega_{S^n}$ and deduce from this that ω_{S^n} is not an exact form on S^n. (Hint: Use Problem (4) to reduce the integration to an open subset of S^n on which spherical coordinates yield a global coordinate chart).