1. Draw sketches of the following varieties in $\mathbb{A}^3$ (with coordinates $x$, $y$, and $z$).

   (a) $z^2 - x^2 - y^2 = 0$

   (b) $y - x^2 = 0.$

   (c) $(y - x^2)(z - 1) = 0$

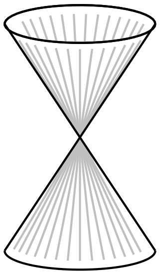
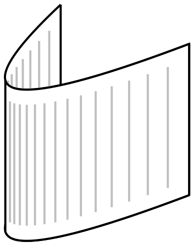
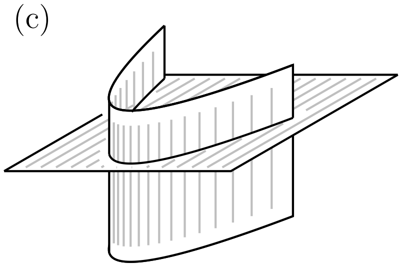
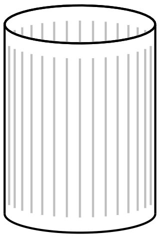
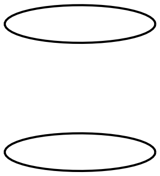
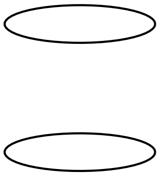
   (d) $x^2 + y^2 - 1 = 0.$

   (e) $x^2 + y^2 - 1 = 0,\ z^2 - 1 = 0.$

   (f) $x^2 + y^2 - 1 = 0,\ z^2 - x^2 - y^2 = 0.$

   (Of course, you only have to draw the real points, i.e, solutions in $\mathbb{R}^3$.)

**Solutions.**

| | | |
|---|---|---|
| (a)<br><br>Cone | (b)<br><br>Stretched Parabola | (c)<br><br>Stretched Parabola union the Plane $z = 1$ |
| (d)<br><br>Cylinder | (e)<br><br>Two Circles | (f)<br><br>The Same Two Circles |

Part (e) is the intersection of (a) and the planes $z = 1$ and $z = -1$, while part (f) is the intersection of (a) and (d). Both intersections are the same. One way to see this without doing the intersections is to notice that the ideals $\langle x^2 + y^2 - 1, z^2 - 1\rangle$ and $\langle x^2 + y^2 - 1, z - x^2 - y^2\rangle$ are equal (see the answer to 2(c)), and hence their zero loci are also equal.

2. (Computing in quotient rings)

(a) Show that $\frac{k[x,y,z]}{\langle x^2-y, x^3-z\rangle} \cong k[x]$.

Recall that a ring $A$ is called a *domain* if whenever $a_1$, $a_2 \in A$ are not zero, then $a_1 \cdot a_2 \neq 0$.

(b) Show that $A = \frac{k[x,y,z]}{\langle (y-x^2)(z-1)\rangle}$ is not a domain.

(c) Is $B = \frac{k[x,y,z]}{\langle x^2+y^2-1, z^2-x^2-y^2\rangle}$ a domain?

Notes: (1) To show that a ring is *not* a domain, you need to find two elements $f_1$ and $f_2$ of $A$ such that $f_1 \neq 0$, $f_2 \neq 0$, but $f_1 f_2 = 0$. Since our rings are rings of functions on algebraic varieties, one way to show that a function is not zero is to evaluate it at a point of the corresponding variety. (2) You have already drawn pictures of the geometric shapes corresponding to the rings in 2(*b*,*c*).

**Solutions.**

(a) Let $J = \langle y - x^2, z - x^3\rangle$. From the definition of $J$ we have $y \equiv x^2 \mod J$ and $z \equiv x^3 \mod J$. Therefore in the quotient ring $k[x,y,z]/J$ we can replace any $y$ by $x^2$ and any $z$ by $x^3$, leaving a polynomial only in $x$. This shows that $k[x,y,z]/J$ is $k[x]$, or possibly smaller, if $J$ also contains a polynomial only in $x$.

To see that the quotient *is* only $k[x]$ and no smaller, one way is to note that if there were a non-zero polynomial $q(x)$ in the ideal $J$, it would imply that all the points $(x,y,z)$ satisfying the equations $y = x^2$ and $z = x^3$ would also satisfy the equation $q(x)$. But $q(x)$ is a polynomial in one variable, so has finitely many roots. I.e., if there were a non-zero $q(x) \in J$ it would imply that there are only finitely many possible $x$-coordinates among the points $(x,y,z)$ satisfying the conditions $y = x^2$ and $z = x^3$. But the points $(t, t^2, t^3)$ with $t \in k$ give infinitely many points satisfying these equations with different $x$-coordinates, and therefore no such $q(x)$ exists. We conclude that $k[x,y,z]/J \cong k[x]$.

**Alternate Solution.** Consider the homomorphism $\psi\colon k[x,y,z] \longrightarrow k[x]$ defined by $\psi(x) = x$, $\psi(y) = x^2$, and $\psi(z) = x^3$. This map is evidently surjective. Let $I = \mathrm{Ker}(\psi)$. Since $\psi(y - x^2) = x^2 - x^2 = 0$ and $\psi(z - x^3) = x^3 - x^3 = 0$ we see that both $y - x^2$ and $z - x^3$ are in $I$, and therefore that $J \subseteq I$. Thus the map $\psi$ factors through the quotient map $\pi\colon k[x,y,z] \longrightarrow k[x,y,z]/J$, i.e,. there exists a map $\varphi\colon k[x,y,z]/J \longrightarrow k[x]$ such that $\psi = \varphi \circ \pi$ :

$$\begin{tikzcd}
k[x,y,z] \arrow[r, "\psi"] \arrow[d, "\pi"'] & k[x] \\
k[x,y,z]/J \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

Since $\psi$ is surjective, so is $\varphi$. To show that $\varphi$ is an isomorphism we then only need to show that $\varphi$ is injective (or, equivalently, that $I \subseteq J$).

Let $A = k[x,y,z]/J$. An element of $A$ is a coset of $J$. By the substitution arguments from part (a), every polynomial in $k[x,y,z]$ is congruent, modulo $J$, to a polynomial $q(x)$ only in $x$. This is the same as saying that the composite map

$$k[x] \xhookrightarrow{i} k[x,y,z] \xrightarrow{\pi} A$$

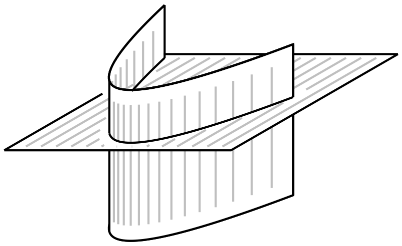
is surjective, where $i$ is the natural inclusion and $\pi$ the quotient map.

The composition $\varphi \circ \pi$ is $\psi$, and $\psi \circ i$ is the map $k[x] \longrightarrow k[x]$ sending $x$ to $x$, i.e., is the identity map. Thus $\varphi \circ (\pi \circ i) = (\varphi \circ \pi) \circ i = \psi \circ i = 1_{k[x]}$. Putting this together we have the maps

$$\begin{tikzcd}
k[x] \arrow[r, two heads, "\pi \circ i"] \arrow[rr, bend left, "1_{k[x]}"] & A \arrow[r, "\varphi"] & k[x]
\end{tikzcd}$$

Since the composition $\varphi \circ (\pi \circ i) = 1_{k[x]}$ is injective (it is the identity map!), and $\pi \circ i$ is surjective, we conclude that $\varphi$ is also surjective. $\square$

(b) The variety cut out by the equation $(y - x^2)(z-1) = 0$ is the one pictured in 1(c).



$X:\ (y - x^2)(z - 1) = 0$

Let $f = y - x^2$ and $g = z - 1$, and let $X$ be the variety $(y - x^2)(z-1) = 0$. Let $\overline{f}$ and $\overline{g}$ be the images of $f$ and $g$ in

$$k[X] = k[x,y,z]/\langle (y - x^2)(z-1) \rangle.$$

From the definition of the quotient, it is clear that $\overline{f}\overline{g} = 0$, so to show that $k[X]$ is not a domain it is sufficient to show that $\overline{f} \neq 0$ and $\overline{g} \neq 0$. Following the suggestion, we look for points on $X$ where $\overline{f}$ and $\overline{g}$ take on nonzero values.

The point $(2,2,1)$ is on $X$ (it is on the plane $z = 1$) and $f(2,2,1) = 2 - 2^2 \neq 0$. This shows that $\overline{f} \neq 0$. The point $(2,4,0)$ is on $X$ (it is on the stretched parabola $y - x^2 = 0$), and $g(2,4,0) = 0 - 1 = -1 \neq 0$, so $\overline{g} \neq 0$. Therefore $k[X]$ is not a domain.

(c) The ring $k[x,y,z]/\langle x^2 + y^2 - 1, z^2 - x^2 - y^2 \rangle$ is not a domain.

As the pictures in 1(e) and 1(f) suggest, the ideals $I := \langle x^2 + y^2 - 1, z^2 - x^2 - y^2 \rangle$ and $J := \langle x^2 + y^2 - 1, z^2 - 1 \rangle$ are equal. This is straightforward to see algebraically. Since $(z^2 - x^2 - y^2) + (x^2 + y^2 - 1) = z^2 - 1$ we see that $z^2 - 1 \in I$ so that $J \subseteq I$.

Conversely since $(z^2-1)-(x^2+y^2-1) = z^2-x^2-y^2$ we see that $z^2-x^2-y^2 \in J$ so that $I \subseteq J$. Therefore $I = J$.

Let $X$ be the affine variety defined by the equations $x^2+y^2-1=0$ and $z^2-1=0$. As question 1(f) shows, $X$ is a union of two disjoint circles. From the equations we see that $k[X] = k[x,y,z]/J$, so that $(\overline{z}-1)(\overline{z}+1) = \overline{z}^2-1 = 0 \in k[X]$. Therefore to show that $k[X]$ is not a domain it is sufficient to show that $\overline{z}-1 \neq 0$ and $\overline{z}+1 \neq 0$ in $k[X]$. We do this by the same method above: finding points of $X$ where the two functions are not zero.

Set $f = \overline{z}-1$ and $g = \overline{z}+1$. The point $(1,0,1)$ is on $X$ (it is on the top circle). Since $g(1,0,1) = 1+1 = 2 \neq 0$, the function $g$ is not zero on $X$. The point $(1,0,-1)$ is also on $X$ (it is on the bottom circle). Since $f(1,0,-1) = -1-1 = -2 \neq 0$ the function $f$ is not zero on $X$. Therefore $k[X]$ is not a domain, as claimed.

3. (Morphisms)

(a) Let $\varphi\colon \mathbb{A}^1 \longrightarrow \mathbb{A}^3$ be given by $\varphi(t) = (t, t^2, t^3)$. Show that the image of $\varphi$ is contained in the variety defined by the equations $y-x^2=0$, $z-x^3=0$.

(b) Describe the ring homorphism from $k[x,y,z]/\langle y-x^2, z-x^3\rangle$ to $k[t]$ given by particular, where do $\overline{x}$, $\overline{y}$, and $\overline{z}$ get sent?) Is $\varphi^*$ surjective? Injective?

(c) Let $X$ be $\{(u,v,w) \,\big|\, u^2+v^2+w^2=1\} \subset \mathbb{A}^3$, and $Y$ the affine variety $\{(x,y,z,w) \,\big|\, xy-zw=0\} \subset \mathbb{A}^4$. Does $\varphi = (1+u, 1-u, v+iw, v-iw)$ induce a map from $X$ to $Y$? (Here $i$ is the square root of $-1$.) If so analyze $\varphi^*$ as in part $(b)$.

**Solutions.**

(a) Let $f = y-x^2$ and $g = z-x^3$. Since $f(t,t^2,t^3) = t^2-(t)^2 = 0$ and $g(t,t^2,t^3) = t^3-(t)^3 = 0$ the image of $\varphi$ lies in the variety defined by the equations $y-x^2=0$ and $z-x^3=0$.

(b) Let $X$ be this variety, with ring of functions $k[x,y,z]/\langle y-x^2, z-x^3\rangle$. The functions $\overline{x}$, $\overline{y}$, and $\overline{z}$ are the restrictions of the coordinate functions to $X$, so pulling back by $\varphi$ we have

$$\varphi^*(\overline{x}) = t,\ \ \varphi^*(\overline{y}) = t^2,\ \text{ and }\ \varphi^*(\overline{z}) = t^3.$$

These formulas show that the map $\varphi^*$ is surjective: the image of $\varphi^*$ contains $t$, and hence any polynomial in $t$, and that is precisely the ring $k[t]$.

The map $\varphi^*$ is also injective. By 2(a) $k[X] \cong k[x]$, and the map $\varphi^*$ on $k[x]$ is simply the map "substitute $x = t$", and induces an isomorphism from $k[x]$ to $k[t]$. (Thus, by previous results from class, $X$ is isomorphic to $\mathbb{A}^1$.)

(c) Since

$$(1-u)(1+u)-(v+iw)(v-iw) = (1-u^2)-(v^2+w^2) = 1-u^2-v^2-w^2 = 0 \in k[X],$$

for every point $(u, v, w) \in X$, $\varphi(u, v, w)$ satisfies the equation defining $Y$, and so $\varphi$ does induce a map from $X$ to $Y$.

The pullbacks of the coordinate functions on $Y$ are

$$\varphi^*(\overline{x}) = 1+u, \;\; \varphi^*(\overline{y}) = 1-u, \;\; \varphi^*(\overline{z}) = v+iw, \;\text{ and }\; \varphi^*(\overline{w}) = v-iw,$$

from which we conclude that $\varphi^*(\overline{x}-1) = u$, $\varphi^*(\frac{1}{2}(\overline{z}+\overline{w})) = v$, and $\varphi^*(\frac{1}{2i}(\overline{z}-\overline{w}) = w$.

In other words, the image of $\varphi^*$ contains $u$, $v$, and $w$. Since these generate $k[X] = k[u, v, w]/(u^2+v^2+w^2-1)$, this means that the map $\varphi^*$ is surjective.

However, the map is not injective: $\varphi^*(\overline{x}+\overline{y}-2) = (1+u)+(1-u)-2 = 0$, and the equation $\overline{x}+\overline{y}-2$ is not zero in $k[Y]$. For instance, the point $(2, 3, 1, 6) \in Y$ and $2+3-2 = 3 \neq 0$. Therefore the map $\varphi^*$ is a surjection but not an injection. In particular, since $\varphi^*$ is not an isomorphism, the map $\varphi$ is not an isomorphism either.