1. Let X and Y be two affine varieties, with rings of functions $R[X]$ and $R[Y]$. In this problem we will use the theorem from the classes of Jan. 17th and 21st to prove that X and Y are isomorphic varieties if and only if $R[X]$ and $R[Y]$ are isomorphic rings.

(a) Explain why $(1_X)^* = 1_{R[X]}$.

Here 1_X and $1_{R[X]}$ are being used in the category-theoretic sense. They are, respectively, the identity morphism $1_X\colon X \longrightarrow X$ and the identity ring homomorphism $1_{R[X]}\colon R[X] \longrightarrow R[X]$.

(b) Suppose that $\varphi\colon X \longrightarrow X$ is a morphism of affine varieties and that $\varphi^* = 1_{R[X]}$. Explain why must have $\varphi = 1_X$.

(c) Suppose that X and Y are isomorphic affine varieties. Writing out the definition of "isomorphic varities" and applying the functor to rings, explain why $R[X]$ and $R[Y]$ are isomorphic rings.

(d) Now suppose that $R[X]$ and $R[Y]$ are isomorphic rings. Write out the definition of "isomorphic rings" and use part (c) of the theorem as well as (b) above to show that X and Y are isomorphic varieties.

Solutions.

(a) By definition of pullback, for any $f \in R[X]$, $(1_X)^* f = f \circ 1_X = f$, so $(1_X)^*(f) = f$ for all $f \in R[X]$. But this is exactly the identity homomorphism $1_{R[X]}$.

(b) By the theorem in class, pullback gives a bijection between maps of varieties and homomorphisms of rings. In part (a) we saw that $(1_X)^* = 1_{R[X]}$. Hence (since the association between morphisms of varieties and ring homomorphisms is one-to-one), if $\varphi\colon X \longrightarrow X$ is a morphism such that $\varphi^* = 1_{R[X]}$ then $\varphi = 1_X$.

Alternatively, we could repeat the idea of the proof of this part of the theorem. Let $X \subseteq \mathbb{A}^n$ with coordinate functions $x_1, \ldots, x_n$. The map φ is determined by the pullback of the coordinate functions; specifically $\varphi = (\varphi^*(x_1), \varphi^*(x_2), \ldots, \varphi^*(x_n))$. By hypothesis we have $\varphi^*(x_i) = 1_{R[X]}(x_i) = x_i$ for all i. Therefore the map φ is given by $\varphi(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n)$, so that $\varphi = 1_X$.

(c) If X and Y are isomorphic varieties, then there are morphisms $\varphi_1\colon X \longrightarrow Y$ and $\varphi_2\colon Y \longrightarrow X$ such that $\varphi_2 \circ \varphi_1 = 1_X$ and $\varphi_1 \circ \varphi_2 = 1_Y$. Applying the functor from affine varieties to rings we get ring homomorphisms $\varphi_1^*\colon R[Y] \longrightarrow R[X]$ and $\varphi_2^*\colon R[X] \longrightarrow R[Y]$ so that

$$\begin{aligned} \varphi_1^* \circ \varphi_2^* &= (\varphi_2 \circ \varphi_1)^* = (1_X)^* = 1_{R[X]}, \text{ and} \\ \varphi_2^* \circ \varphi_1^* &= (\varphi_1 \circ \varphi_2)^* = (1_Y)^* = 1_{R[Y]}. \end{aligned}$$

Therefore $R[X]$ and $R[Y]$ are isomorphic rings.

(d) Now suppose that $R[X]$ and $R[Y]$ are isomorphic rings. By definition this means that there are ring homomorphisms $\psi_1\colon R[Y] \longrightarrow R[X]$ and $\psi_2\colon R[X] \longrightarrow R[Y]$ so that $\psi_1 \circ \psi_2 = 1_{R[X]}$ and $\psi_2 \circ \psi_1 = 1_{R[Y]}$.

By part (c) of the Theorem from the Jan. 17th class, there are morphisms $\varphi_1\colon X \longrightarrow Y$ and $\varphi_2\colon Y \longrightarrow X$ so that $\varphi_1^* = \psi_1$ and $\varphi_2^* = \psi_2$.

Consider the map $\varphi_2 \circ \varphi_1$. Applying the functor to rings we get

$$(\varphi_2 \circ \varphi_1)^* = \varphi_1^* \circ \varphi_2^* = \psi_1 \circ \psi_2 = 1_{R[X]}.$$

By part (b) of this question, that means that $\varphi_2 \circ \varphi_1 = 1_X$. Similarly, since

$$(\varphi_1 \circ \varphi_2)^* = \varphi_2^* \circ \varphi_1^* = \psi_2 \circ \psi_1 = 1_{R[Y]},$$

we conclude that $\varphi_1 \circ \varphi_2 = 1_Y$. Thus φ_1 and φ_2 are isomorphisms, and so X and Y are isomorphic.

2. In this question we will see an example of a morphism of affine varieties which is one-to-one on points, but which is not an isomorphism. (In other words, in the category of affine varieties, isomorphism implies more than just one-to-one.) Let $X = \mathbb{A}^1$ with ring of functions $k[t]$, and let Y be the subset of $\mathbb{A}^2$ given by the equation $y^2 = x^3$.

(a) Let $\varphi\colon X \longrightarrow \mathbb{A}^2$ be the map given by $\varphi(t) = (t^2, t^3)$. Show the image of φ lies in Y, so that φ defines a morphism $\varphi\colon X \longrightarrow Y$.

(b) Show that φ is surjective. (i.e., given $(x, y) \in Y$, show that there is a t such that $\varphi(t) = (x, y)$.)

(c) Show that φ is injective.

(d) Draw a sketch of Y ($\mathbb{R}^2$ points only). One suggestion: from part (b) you know that Y is the image of φ, so you can use the parameterization given by φ to see what Y looks like.

(e) Compute the image of the ring homomorphism $\varphi^*\colon R[Y] \longrightarrow R[X]$ (and recall that $R[X] = k[t]$). Is φ^* surjective?

(f) Explain why φ cannot be an isomorphism of affine varieties.

Solutions.

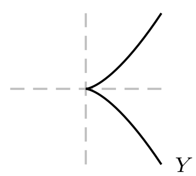
(a) Let $g := y^2 - x^3$ be the equation defining Y. Since $g(\varphi(t)) = g(t^2, t^3) = (t^3)^2 - (t^2)^3 = t^6 - t^6 = 0$, we see that the image of φ lies in Y.

(b) Let (x, y) be a point of Y. If $x = 0$ then the equation $y^2 - x^3 = 0$ implies that $y = 0$ too, and then $\varphi(0) = (0, 0) = (x, y)$ is in the image of φ. We may therefore assume that $x \neq 0$.

In this case, set $t = y/x$. Then $\varphi(t) = (y^2/x^2, y^3/x^3)$. But from $y^2 = x^3$ we deduce that $y^2/x^2 = x^3/x^2 = x$, and that $y^3/x^3 = y(y^2/x^3) = y(1) = y$. Therefore $\varphi(t) = (x, y)$, so that φ is surjective.

(c) Suppose that $\varphi(t_1) = \varphi(t_2)$, i.e., that $(t_1^2, t_1^3) = (t_2^2, t_2^3)$. If $t_1^2 = 0$ then $t_2^2 = 0$ and therefore both $t_1 = 0$ and $t_2 = 0$, i.e., $t_1 = t_2$. We may therefore assume that neither t_1^2 nor t_2^2 are zero. But then $t_1 = (t_1^3/t_1^2) = (t_2^3/t_2^2) = t_2$. Therefore φ is injective.

(d) Using the parameterization φ, we see that the real points of Y look like this:



(e) From the formula for φ we see that $\varphi^*(x) = t^2$ and $\varphi^*(y) = t^3$. Thus a polynomial $f = \sum c_{ij} x^i y^j$ in $R[Y]$ pulls back to $\varphi^*(f) = \sum c_{ij} t^{2i+3j}$. In particular, since there is no way to write 1 as a sum $2i + 3j$ with both $i, j \geqslant 0$, we see that $t = t^1$ is not in the image of φ^*. Therefore φ^* is not surjective.

Remark. It is not hard to check that any natural number greater than 1 can be written in the form $2i + 3j$ with $i, j \geqslant 0$. It then follows that a polynomial $g \in k[t]$ is in the image of φ^* if and only if the coefficient of t in g is zero.

(f) By the result from question 1 above, φ is an isomorphism of varieties if and only if φ^* is an isomorphism of rings. Since φ^* is not surjective, it is not an isomorphism of rings. Therefore φ is not an isomorphism of varieties.

Remark. Why isn't a bijection of varieties an isomorphism? The definition of isomorphism from category theory helps us: We need there to be a map $\varphi_2 \colon Y \longrightarrow X$ of varieties so that the composition of φ and φ_2 are the identity maps (of X and Y, depending on the order of composition). Since φ is a bijection of sets, there is certainly a map φ_2 of *sets* from Y to X which undoes φ. The problem is that we cannot give such a map with algebraic functions. The real problem is that we cannot undo φ in a neighbourhood of $(0,0) \in Y$. As the picture in (d) shows, something funny is going on at $(0,0) \in Y$, making it look different from $\mathbb{A}^1$ there.

3. Consider the following four affine varieties, all contained in $\mathbb{A}^3$.

$$\begin{aligned} X &= \left\{(x_1,x_2,x_3) \,\middle|\, x_1^2 + x_2^2 - 1 = 0\right\} \subset \mathbb{A}^3 \\ Y &= \left\{(y_1,y_2,y_3) \,\middle|\, y_1^2 + y_2^2 - y_3^2 = 0\right\} \subset \mathbb{A}^3 \\ Z &= \left\{(z_1,z_2,z_3) \,\middle|\, z_1^2 + z_2^2 + z_3^2 - 625 = 0\right\} \subset \mathbb{A}^3 \\ W &= \left\{(w_1,w_2,w_3) \,\middle|\, w_1^2 + w_2^2 - w_3 = 0\right\} \subset \mathbb{A}^3 \end{aligned}$$

(a) Draw sketches of X, Y, Z, and W.

Define a map $\varphi_1 \colon X \longrightarrow \mathbb{A}^3$ by $\varphi_1(x_1,x_2,x_3) = (x_1x_3, x_2x_3, x_3)$.

(b) Is the image of φ_1 contained in Y, Z, or W? (Justify your answer.)

Define a map $\varphi_2 \colon X \longrightarrow \mathbb{A}^3$ by $\varphi_2(x_1,x_2,x_3) = (-9x_1 + 12x_2, 12x_1 - 16x_2, 20x_1 + 15x_2)$.

(c) Is the image of φ_2 contained in Y, Z, or W? (Justify your answer.)

Define a map $\varphi_3 \colon Y \longrightarrow \mathbb{A}^3$ by $\varphi_3(y_1,y_2,y_3) = (y_1, y_2, y_3^2)$.

(d) Is the image of φ_3 contained in X, Z, or W? (Justify your answer.)

One of the maps (b)–(d) has image in W.

(e) What is the pullback of $3\overline{w}_1 - \overline{w}_2^2 + \overline{w}_3 \in R[W]$ under this map?

Now we will try and go the other way, from a map of rings to a map of varieties. Define a ring homomorphism

$$R[X] = \frac{k[x_1,x_2,x_3]}{(x_1^2 + x_2^2 - 1)} \longleftarrow \frac{k[w_1,w_2,w_3]}{(w_1^2 + w_2^2 - w_3)} = R[W] \colon \psi$$

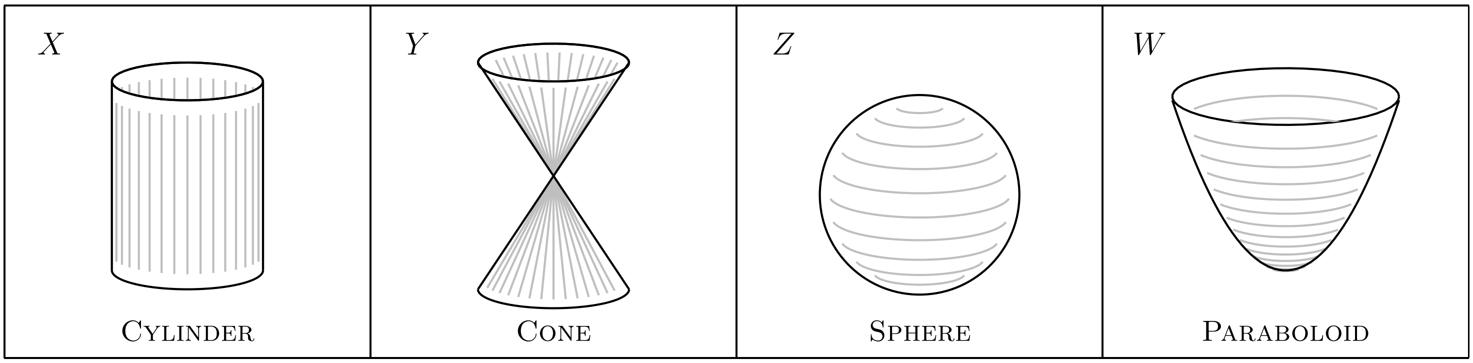
by the rule $\psi(\overline{w}_1) = 2\overline{x}_1$, $\psi(\overline{w}_2) = 2\overline{x}_2$, $\psi(\overline{w}_3) = 4$.

(f) Check that this ring homomorphism is well-defined by showing that $\psi(\overline{w}_1^2 + \overline{w}_2^2 - \overline{w}_3) = 0$.

(g) What geometric map $\varphi \colon X \longrightarrow W$ does the ring homomorphism ψ correspond to? (Write your formula for φ in the form $\varphi(x_1,x_2,x_3) =$ (formulas in x_1, x_2, x_3) $\subset \mathbb{A}^3$ as in (b)–(d) above.)

Solution.

(a)



To check the image of each map φ_j in (b)–(d) we see if the formulas for φ_j satisfy the equations of X, Y, Z, or W. Equivalently, we see if the pullbacks of the equations for X, Y, Z, or W under φ_j^* are zero.

(b) The image of φ_1 lies in Y, since applying the equation $y_1^2+y_2^2-y_3^2$ to (x_1x_3, x_2x_3, x_3), and using the fact that $x_1^2+x_2^2=1$ we get

$$(x_1x_3)^2+(x_2x_3)^2-(x_3)^2 = x_3^2(x_1^2+x_2^2)-x_3^2 = x_3^2-x_3^2 = 0$$

and thus for every $(x_1,x_2,x_3)\in X$, $\varphi_1(x_1,x_2,x_3)$ lies in Y.

(c) The image of φ_2 lies in Z, since for every $(x_1,x_2,x_3)\in X$, applying the equation $z_1^2+z_2^2+z_3^2-625$ to $\varphi_2(x_1,x_2,x_3) = (-9x_1+12x_2, 12x_1-16x_2, 20x_1+15x_2)$ gives

$$\begin{aligned}
&(-9x_1+12x_2)^2+(12x_1-16x_2)^2+(20x_1+15x_2)^2-625\\
&\quad= (81x_1^2-216x_1x_2+144x_2^2)+(144x_1^2-384x_1x_2+256x_2^2)+(400x_1^2+600x_1x_2+225x_2^2)-625\\
&\quad= (81+144+400)x_1^2+(-216-384+600)x_1x_2+(144+256+225)x_2^2-625\\
&\quad= 625x_1^2+0x_1x_2+625x_2^2-625=0.
\end{aligned}$$

(d) The image of φ_3 lies in W; for every $(y_1,y_2,y_3)\in Y$, applying the equation $w_1^2+w_2^2-w_3$ to $\varphi_3(y_1,y_2,y_3) = (y_1,y_2,y_3^2)$ gives

$$(y_1)^2-(y_2)^2-y_3^2 = y_1^2+y_2^2-y_3^2=0,$$

since this last condition is the defining equation of Y.

(e) The map φ_3 has image in W. The definition of $\overline{w}_1$, $\overline{w}_2$, and $\overline{w}_3$ is that they are the restriction of the coordinate functions w_1, w_2, and w_3 to W. The definition of pullback by φ_3 is composition with φ_3. Since φ_3 is given by the formula $\varphi_3(y_1,y_2,y_3)=(y_1,y_2,y_3^2)$, we have

$$\varphi_3^*(\overline{w}_1)=\overline{y}_1, \varphi_3^*(\overline{w}_2)=\overline{y}_2, \text{ and } \varphi_3^*(\overline{w}_3)=\overline{y}_3^2.$$

Finally, using the fact that φ_3^* is a ring homomorphism we get

$$\varphi_3^*(3\overline{w}_1 - \overline{w}_2^2 + \overline{w}_3) = 3\overline{y}_1 - \overline{y}_2^3 - \overline{y}_3^2.$$

(f) By the definition of ψ we have

$$\psi(\overline{w}_1^2 + \overline{w}_2^2 - \overline{w}_2) = (2\overline{x}_1)^2 + (2\overline{x}_2)^2 - 4 = 4(\overline{x}_1^2 + \overline{x}_2^2) - 4 = 4 \cdot 1 - 4 = 0.$$

(g) In general, as in (e), the pullbacks of the coordinate functions show the formulas used to define the map. From the pullbacks in (f) we conclude that the geometric map $\varphi\colon X \longrightarrow W$ corresponding to ψ (i.e, so that $\psi = \varphi^*$) is $\varphi(x_1, x_2, x_3) = (2x_1, 2x_2, 4)$.

Geometrically this map collapses the cylinder to a circle (by ignoring the value of the x_3 coordinate), and then puts that circle into the paraboloid as a circle of radius 4 at height 4.