1. In this problem we will use parts of the algebra-geometry correspondence that we have built up to prove the following result in commutative algebra:

> Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal, and $\overline{I}$ the intersection of all the maximal ideals of $k[x_1, \ldots, x_n]$ containing I. Then $\overline{I} = \sqrt{I}$.

(a) Show that a maximal ideal is a radical ideal. (Suggestion: It may help to rewrite the condition that $I \subset A$ is a radical ideal in terms of the quotient ring A/I.)

(b) Show that an arbitrary intersection of maximal ideals is a radical ideal. (You can use results from the previous homework.)

Now assume that $J \subseteq k[x_1, \ldots, x_n]$ is a radical ideal, and let $\overline{I}$ be the intersection of all maximal ideals containing J. I.e., $\overline{I} = \bigcap_{J \subseteq \mathfrak{m}} \mathfrak{m}$, where each $\mathfrak{m}$ is a maximal ideal.

(c) Show that every maximal ideal containing J also contains $\overline{I}$.

(d) Show that $J \subseteq \overline{I}$.

(e) Show that every maximal ideal containing $\overline{I}$ also contains J.

Next, using parts of the algebra-geometry dictionary we have seen in class:

(f) Explain why $V(J) = V(\overline{I})$. (Suggestion: what do (c) and (e) say about the points of $V(J)$ and $V(\overline{I})$?)

(g) Explain why we then know that $J = \overline{I}$.

Finally, let $I \subset k[x_1, \ldots, x_n]$ be any ideal, and set $J = \sqrt{I}$.

(h) Show that any maximal ideal containing I also contains J.

(i) Show that any maximal ideal containing J also contains I.

(j) Prove the commutative algebra statement above.

Solutions.

(a) Let A be a ring and $I \subseteq A$ an ideal. The condition that I is a radical ideal is that for any $f \in A$, if $f^n \in I$ for some $n \geqslant 1$ then $f \in I$. In terms of the quotient, this means that for any $\overline{f} \in A/I$, if $\overline{f}^n = 0$ for some $n \geqslant 1$ then $\overline{f} = 0$.

If I is a maximal ideal, then A/I is a field and hence a domain. Therefore if $\overline{f}^n = 0$ we must have $\overline{f} = 0$, so I is a radical ideal. (More generally, if I is a prime ideal then A/I is a domain, and this argument gives another way of showing that a prime ideal is radical.)

(b) The argument from **H4**, **Q2(d)** works essentially without change. Suppose that J_α, $\alpha \in S$ is a collection of radical ideals in a ring A, and set $\overline{I} = \cap_{\alpha\in S} J_\alpha$. Suppose that $f \in A$ and that $f^n \in \overline{I}$ for some $n \geqslant 1$. Then $f^n \in J_\alpha$ for each $\alpha \in S$. Since each J_α is a radical ideal, this means that $f \in J_\alpha$ for each α. Therefore $f \in \cap_{\alpha\in S} J_\alpha = \overline{I}$, and so $\overline{I}$ is a radical ideal.

(c) The intersection of sets is always contained in each of the sets being intersected. By definition $\overline{I}$ is the intersection of all maximal ideals containing J, and hence is contained in each maximal ideal containing J.

(d) If $\mathfrak{m}$ is a maximal ideal containing J, then certainly $J \subseteq \mathfrak{m}$, and so J will also be contained in the intersection of all such maximal ideals. Since that intersection is $\overline{I}$, $J \subseteq \overline{I}$.

(e) By part (d) we have $J \subseteq \overline{I}$. Therefore if $\mathfrak{m}$ is a maximal ideal containing $\overline{I}$, we have $J \subseteq \overline{I} \subseteq \mathfrak{m}$, so $\mathfrak{m}$ also contains J.

(f) Part (c) tells us that every maximal ideal containing J also contains $\overline{I}$, and part (e) that every maximal ideal containing $\overline{I}$ contains J. Thus the set of maximal ideals containing $\overline{I}$ and the set of maximal ideals containing J are the same.

Maximal ideals in $k[x_1, \ldots, x_n]$ correspond to points of $\mathbb{A}^n$. By the order reversing correspondence between ideals and varieties, the points of $V(J)$ correspond to the maximal ideals containing J, and the points of $V(\overline{I})$ correspond to the maximal ideals containing $\overline{I}$. By the previous discussion these sets of maximal ideals are the same, and therefore the sets of points are the same. Thus the points of $V(J)$ and $V(\overline{I})$ are the same, and so $V(J) = V(\overline{I})$.

(g) The ideal/variety correspondence gives a bijection between subvarieties of $\mathbb{A}^n$ and radical ideals of $k[x_1, \ldots, x_n]$. By part (b) $\overline{I}$ is a radical ideal, and J is a radical ideal by assumption. Since $V(J) = V(\overline{I})$ the correspondence then tells us that $J = \overline{I}$.

(h) Let $\mathfrak{m}$ be a maximal ideal containing I, and f any element of J. By definition there is an $n \geqslant 1$ so that $f^n \in I$, and therefore $f^n \in \mathfrak{m}$ too. By part (a), $\mathfrak{m}$ is a radical ideal, so we have $f \in \mathfrak{m}$. Thus every element of J is an element of $\mathfrak{m}$ and so $J \subseteq \mathfrak{m}$.

(i) We have already seen in class that $I \subseteq \sqrt{I} = J$, so any maximal ideal containing J also contains I.

(j) Let $I \subseteq k[x_1, \ldots, x_n]$ be any ideal, and set $J = \sqrt{I}$. By parts (c)–(g) we have that $\bigcap_{J \subseteq \mathfrak{m}} \mathfrak{m} = J$. By parts (h) and (i) we have that the maximal ideals containing I are the same as the maximal ideals containing J. Thus $\bigcap_{I \subseteq \mathfrak{m}} \mathfrak{m} = \bigcap_{J \subseteq \mathfrak{m}} \mathfrak{m} = J = \sqrt{I}$.

2. In this question we will explore the construction of sum of ideals. Given a ring A, and a (possibly infinite) collection of ideals $I_\alpha \subset A$, $\alpha \in S$ recall that we have defined $\sum_{\alpha \in S} I_\alpha$ as all possible finite sums of elements in the I_α, i.e.,

$$\sum_{\alpha \in S} I_\alpha = \left\{ f_{\alpha_1} + f_{\alpha_2} + \cdots + f_{\alpha_k} \,\middle|\, f_{\alpha_j} \in I_{\alpha_j} \right\}.$$

(a) Show that $\sum_{\alpha \in S} I_\alpha$ is an ideal.

(b) Suppose that A is a Noetherian ring. Show that there is a finite subset $S' \subseteq S$ such that $\sum_{\alpha \in S'} I_\alpha = \sum_{\alpha \in S} I_\alpha$.

(c) Suppose that X is an affine variety with ring of functions $R[X]$. Let Z_α, $\alpha \in S$ be a collection of closed subsets of X corresponding to ideals J_α, $\alpha \in S$. Show that

$$V(\sum_{\alpha \in S} J_\alpha) = \bigcap_{\alpha \in S} Z_\alpha$$

as claimed in class.

Solution.

(a) Suppose that $f \in \sum_{\alpha \in S} I_\alpha$, so that $f = f_{\alpha_1} + \cdots + f_{\alpha_k}$ for some $\alpha_1, \ldots, \alpha_k \in S$. Let g be any element of A. Then $g f_{\alpha_i} \in I_{\alpha_i}$ for $i = 1, \ldots, k$, since each I_{α_i} is an ideal. Therefore $gf = (g f_{\alpha_1}) + \cdots + (g f_{\alpha_k}) \in \sum_{\alpha \in S} I_\alpha$.

Now suppose that $g \in \sum_{\alpha \in S}$, so that $g = g_{\beta_1} + \cdots + g_{\beta_\ell}$ with $g_{\beta_j} \in I_{\beta_j}$ for some $\beta_1, \ldots, \beta_\ell \in S$. It is easier notationally to be able to assume that the index sets for the terms of f and g are the same. We can do this by taking the union of the two index sets, say $\{\alpha_1, \ldots, \alpha_k\} \cup \{\beta_1, \ldots, \beta_\ell\} = \{\gamma_1, \ldots, \gamma_r\}$, and setting $f_{\gamma_j} = 0$

whenever γ_j is not one of the α's, and similarly $g_{\gamma_j} = 0$ whenever γ_j is not one of the β's. Then

$$f + g = (f_{\gamma_1} + g_{\gamma_1}) + \cdots + (f_{\gamma_r} + g_{\gamma_r}).$$

Each $f_{\gamma_j} + g_{\gamma_j} \in I_{\gamma_j}$ since I_{γ_j} is an ideal. Thus $f + g \in \sum_{\alpha \in S} I_\alpha$ so $\sum_{\alpha \in S} I_\alpha$ is an ideal.

(b) Consider the set T of ideals of A of the form $\sum_{\alpha \in S'} I_\alpha$ with S' a finite subset of S. Since A is Noetherian, T has a maximal element, i.e., there is an S' such that the ideal $I = \sum_{\alpha \in S'} I_\alpha$ is not strictly contained in any other element of T.

The claim is that $I = \sum_{\alpha \in S} I_\alpha$. If not, then there is some element f of $\sum_{\alpha \in S} I_\alpha$ which is not in I. Let S'' be the indices appearing when writing f out as a sum of elements in the I_α, and set $S''' = S' \cup S''$. Then the ideal $\sum_{\alpha \in S'''} I_\alpha$ is in T, contains I, and contains f. I.e., this is an ideal in T which strictly contains I, contradicting the maximality of I. Therefore $I = \sum_{\alpha \in S} I_\alpha$.

(c) Suppose that $z \in \cap_{\alpha \in S} Z_\alpha$. Then for any $f_\alpha \in J_\alpha$, $f_\alpha(z) = 0$. It follows that for any sum $f = f_{\alpha_1} + \cdots + f_{\alpha_k}$ with each $f_{\alpha_j} \in I_{\alpha_j}$ we have $f(z) = 0$. Thus $\cap_{\alpha \in S} Z_\alpha \subseteq V(\sum_{\alpha \in S} J_\alpha)$.

To see the opposite inclusion, note that for any $\beta \in S$, $J_\beta \subseteq \sum_{\alpha \in S} J_\alpha$, and hence by the order reversing correspondence between ideals and subvarieties,

$$V\left(\sum_{\alpha \in S} J_\alpha\right) \subseteq V(J_\beta) = Z_\beta.$$

Since this is true for all $\beta \in S$, we have the inclusion $V(\sum_\alpha J_\alpha) \subseteq \cap_{\alpha \in S} Z_\alpha$, and thus $V(\sum_{\alpha \in S} J_\alpha) = \cap_{\alpha \in S} Z_\alpha$.

3. The *elementary symmetric polynomials* in x_1, x_2, and x_3 are the polynomials $e_1 = x_1 + x_2 + x_3$, $e_2 = x_1x_2 + x_2x_3 + x_1x_3$, and $e_3 = x_1x_2x_3$. It is a useful result in algebra that these polynomials are algebraically independent over any field. This means that for any polynomial $f(y_1, y_2, y_3) \in k[y_1, y_2, y_3]$ the polynomial $f(e_1, e_2, e_3) \in k[x_1, x_2, x_3]$ is zero only if f was zero to start with.

In contrast, the functions $g_1 = x_1^2$, $g_2 = x_1x_2$, and $g_3 = x_2^2$ are not algebraically independent. Letting $f(y_1, y_2, y_3) = y_1y_3 - y_2^2$, we have $f \neq 0$ but $f(g_1, g_2, g_3) = 0$.

In this problem we will use combination of geometric and algebraic arguments (and thus the algebra $\longleftrightarrow$ geometry dictionary) to show that e_1, e_2, and e_3 are algebraically independent.

(a) Suppose that $\varphi\colon X \longrightarrow Y$ is a morphism of affine varieties, and that φ is surjective. Show that the homomorphism $\varphi^*\colon R[Y] \longrightarrow R[X]$ is injective.

(b) Let $X = \mathbb{A}^3$ with ring of functions $k[x_1, x_2, x_3]$, and let Y also be $\mathbb{A}^3$ with ring of functions $k[y_1, y_2, y_3]$. Let $\varphi \colon X \longrightarrow Y$ be the map

$$\varphi(x_1, x_2, x_3) = \Big(x_1 + x_2 + x_3,\ x_1x_2 + x_2x_3 + x_1x_3,\ x_1x_2x_3\Big).$$

So, for instance, $\varphi(3,\ 1,\ 5) = (3 + 1 + 5,\ 3 \cdot 1 + 1 \cdot 5 + 3 \cdot 5,\ 3 \cdot 1 \cdot 5) = (9,\ 23,\ 15)$.

Describe the pullback map φ^*. In particular, what are $\varphi^*(y_1)$, $\varphi^*(y_2)$, and $\varphi^*(y_3)$?

(c) Expand the product $(t - \alpha)(t - \beta)(t - \gamma)$.

(d) For any $(a, b, c) \in Y$, consider the polynomial $t^3 - at^2 + bt - c$ and let α, β, and γ be the roots. Show that $\varphi(\alpha, \beta, \gamma) = (a, b, c)$.

(e) Prove that e_1, e_2, and e_3 are algebraically independent.

Solution.

(a) Suppose that $f \in R[Y]$ is in the kernel of φ^*. Then $\varphi^*(f)$ is the zero function on X, so $\varphi^*(f)(x) = f(\varphi(x)) = 0$ for all $x \in X$. Let y be any point of Y. Since φ is surjective, there is an $x \in X$ such that $\varphi(x) = y$. By the previous calculation, this means that $f(y) = f(\varphi(x)) = 0$. In other words, $f(y) = 0$ for all $y \in Y$, so f is the zero function. Thus φ^* is injective.

(b) The pullback map is composition, so

$$\begin{aligned}
\varphi^*(y_1)(x_1, x_2, x_3) &= y_1(\varphi((x_1, x_2, x_3)) = y_1((x_1 + x_2 + x_3,\ x_1x_2 + x_2x_3 + x_1x_3,\ x_1x_2x_3) \\
&= x_1 + x_2 + x_2, \\
\varphi^*(y_2)(x_1, x_2, x_3) &= y_2(\varphi((x_1, x_2, x_3)) = y_2((x_1 + x_2 + x_3,\ x_1x_2 + x_2x_3 + x_1x_3,\ x_1x_2x_3) \\
&= x_1x_2 + x_2x_3 + x_1x_3, \\
\varphi^*(y_3)(x_1, x_2, x_3) &= y_3(\varphi((x_1, x_2, x_3)) = y_3((x_1 + x_2 + x_3,\ x_1x_2 + x_2x_3 + x_1x_3,\ x_1x_2x_3) \\
&= x_1x_2x_3.
\end{aligned}$$

In general, for a polynomial $f(y_1, y_2, y_3) \in R[Y] = k[y_1, y_2, y_3]$ this implies that $\varphi^*(f) = f(x_1 + x_2 + x_3,\ x_1x_2 + x_2x_3 + x_1x_3,\ x_1x_2x_3) = f(e_1, e_2, e_3)$.

(c) $(t - \alpha)(t - \beta)(t - \gamma) = t^3 - (\alpha + \beta + \gamma)t^2 + (\alpha\beta + \beta\gamma + \alpha\gamma)t - \alpha\beta\gamma$.

(d) If α, β, and γ are the roots of $t^3 - at^2 + bt - c$ then

$$t^3 - at^2 + bt - c = (t - \alpha)(t - \beta)(t - \gamma) = t^3 - (\alpha + \beta + \gamma)t^2 + (\alpha\beta + \beta\gamma + \alpha\gamma)t - \alpha\beta\gamma,$$

so $a = \alpha + \beta + \gamma$, $b = \alpha\beta + \beta\gamma + \alpha\gamma$, and $c = \alpha\beta\gamma$. Therfore $\varphi(\alpha, \beta, \gamma) = (a, b, c)$.

(e) From part (b), for any $f(y_1, y_2, y_3) \in k[y_1, y_2, y_3]$, $\varphi^* f = f(e_1, e_2, e_3)$. By part (d) the map φ is surjective, so by (a) this means that φ^* is injective. Thus, the only polynomial $f(y_1, y_2, y_3) \in k[y_1, y_2, y_3]$ such that $f(e_1, e_2, e_3) = 0$ is the zero polynomial. By definition this means that e_1, e_2, and e_3 are algebraically independent.