1. Let $X = \mathbb{A}^n$ and for $2 \leqslant s \leqslant n$ let $V_s$ be the open subset of $X$ which is the complement of the linear space $x_1 = x_2 = x_3 = \cdots = x_s = 0$. Compute (analogously to the computation for $\mathbb{A}^2$ and $s = 2$) the ring of functions $\mathcal{O}_X(V_s)$. (You can make your life easier in the case $s > 2$ by appealing to your answer for $s = 2$.)

**Solution.** First consider the case that $s = 2$. Then $V_2$ is covered by the two principal open sets $x_1 \neq 0$ and $x_2 \neq 0$ with respective coordinate rings $k[x_1, \dots, x_n, \frac{1}{x_1}]$ and $k[x_1, \dots, x_n, \frac{1}{x_2}]$. By construction, an element of $\mathcal{O}_{\mathbb{A}^n}(V_2)$ is a pair $g_1 \in \mathcal{O}_{\mathbb{A}^n}(U_{x_1})$ and $g_2 \in \mathcal{O}_{\mathbb{A}^n}(U_{x_2})$ which agree on the intersection. Write $g_1$ as a polynomial in $x_1$ and $x_2$ whose coefficients are in $k[x_3, x_4, \dots, x_n]$, i.e. as

$$g_1 = \sum b_{ij}(x_3, \cdots, x_n) x_1^i x_2^j$$

with $b_{ij}(x_2, \dots, x_n) \in k[x_3, \dots, x_n]$, $j \geqslant 0$ and $i \in \mathbb{Z}$. Similarly we can write

$$g_2 = \sum c_{ij}(x_3, \cdots, x_n) x_1^i x_2^j$$

with $c_{ij}(x_2, \dots, x_n) \in k[x_3, \dots, x_n]$, $i \geqslant 0$ and $j \in \mathbb{Z}$.

In order for $g_1$ and $g_2$ to agree in $\mathcal{O}_{\mathbb{A}^n}(U_{x_1 x_2}) = k[x_1, \dots, x_n, \frac{1}{x_1}, \frac{1}{x_2}]$ the coefficients of each monomial $x_1^i x_2^j$ must agree. That is, we must have $b_{ij} = c_{ij}$ for all $i$ and $j$. Since $c_{ij} = 0$ when $i < 0$, we have $b_{ij} = 0$ for negative $i$ as well. Thus $b_{ij} \neq 0$ only for nonnegative $i$ and $j$. By the equality $c_{ij} = b_{ij}$ the same is true for $c_{ij}$. Thus both $g_1$ and $g_2$ are polynomials in $x_1, \dots, x_n$ (and the same polynomial). Therefore $\mathcal{O}_{\mathbb{A}^n}(V_2) = k[x_1, \dots, x_n]$.

To deal with the case $s > 2$, we could repeat this type of computation, or take a short-cut. We have $V_2 \subset V_3 \subset V_4 \subset \cdots \subset V_n$. Hence, when $s > 2$ we have an inclusion $V_2 \subset V_s$, and therefore a restriction map $\mathcal{O}_{\mathbb{A}^n}(V_s) \longrightarrow \mathcal{O}_{\mathbb{A}^n}(V_2)$. We also have a restriction map $\mathcal{O}_{\mathbb{A}^n}(\mathbb{A}^n) \longrightarrow \mathcal{O}_{\mathbb{A}^n}(V_s)$. By the first part of this problem, the composite map

$$k[x_1, \dots, x_n] = \mathcal{O}_{\mathbb{A}^n}(\mathbb{A}^n) \longrightarrow \mathcal{O}_{\mathbb{A}^n}(V_s) \longrightarrow \mathcal{O}_{\mathbb{A}^n}(V_2) = k[x_1, \dots, x_n]$$

is an isomorphism. Since $\mathbb{A}^n$ is a domain, the restriction map $\mathcal{O}_{\mathbb{A}^n}(V_s) \longrightarrow \mathcal{O}_{\mathbb{A}^n}(V_2)$ is an inclusion. By the above composition map, above, the map $\mathcal{O}_{\mathbb{A}^n}(V_s) \longrightarrow \mathcal{O}_{\mathbb{A}^n}(V_2)$ is a surjection as well. Thus the map $\mathcal{O}_{\mathbb{A}^n}(V_s) \longrightarrow \mathcal{O}_{\mathbb{A}^n}(V_2)$ is an isomorphism, and so $\mathcal{O}_{\mathbb{A}^n}(V_s) = \mathcal{O}_{\mathbb{A}^n}(V_2) = k[x_1, \dots, x_n]$. $\square$

2. Let $X$ be the affine variety described by the equation $xy - z^2 = 0$ in $\mathbb{A}^3$, and let $U \subset X$ be the complement of $(0, 0, 0) \in X$. In this problem we will compute $\mathcal{O}_X(U)$ and see that it is equal to $R[X]$.

The variety $X$ is covered by the principal open sets $U_x$ and $U_y$, with coordinate rings $k[x,y,z,1/x]/(xy-z^2) \cong k[x,1/x,z]$ and $k[x,y,z,1/y]/(xy-z^2) \cong k[y,1/y,z]$ respectively. Any function $g_1 \in R[U_x]$ can be written as a finite sum $g_1 = \sum a_{ij}x^i z^j$ and any function $g_2 \in R[U_y]$ can be written as a finite sum $g_2 = \sum b_{k\ell}y^k z^\ell$.

(a) What range of indices are valid in the expressions for $g_1$ and $g_2$ above?

We want to look at pairs $(g_1, g_2)$ which agree on $U_x \cap U_y$. The expressions for $g_1$ and $g_2$ above are with respect to different variables. To compare them we need to write them in terms of the same variables.

(b) Use the relation $y = \frac{z^2}{x}$ (valid on $U_x$, and hence also on $U_x \cap U_y$) to write $g_2$ in terms of the variables $x$ and $z$.

(c) In order for $g_1$ to be equal to $g_2$, what must be the relation between the $a_{ij}$ and the $b_{k\ell}$?

(d) Considering the restrictions on the indices from part (a), your formula from (c) will imply additional restrictions on $i$ and $j$. What are they?

(e) For each $i$ and $j$ satisfying the conditions above, show that there is a monomial $x^p y^q z^r$ which is equal to $x^i z^j$ on $U_x$.

(f) Explain why this means that the restriction homomorphism $R[X] \longrightarrow \mathcal{O}_X(U)$ is surjective.

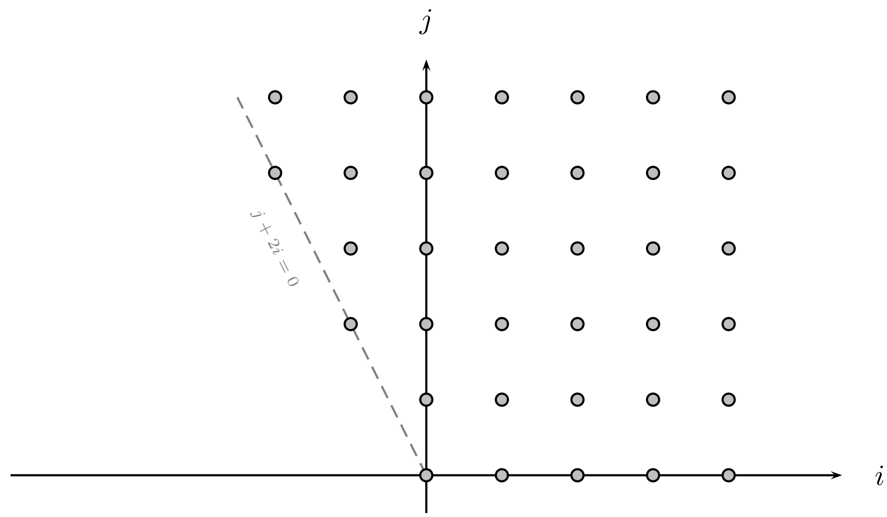
**Solution.**

(a) The ranges are $i \in \mathbb{Z}$ and $j \geqslant 0$ for $g_1$ and $k \in \mathbb{Z}$ and $\ell \geqslant 0$ for $g_2$.

(b) $g_2 = \sum b_{k\ell}y^k z^\ell = \sum b_{k\ell}\left(\frac{z^2}{x}\right)^k z^\ell = \sum b_{k\ell}x^{-k}z^{\ell+2k}$.

(c) In order for $g_1$ and $g_2$ to agree, the coefficients of each monomial must match up. Thus we must have $b_{k\ell} = a_{-k,\ell+2k}$ for all $k$ and $\ell$, or reversing the formula, that $a_{ij} = b_{-i,j+2i}$.

(d) Since $b_{k\ell} = 0$ when $\ell < 0$, we have $a_{ij} = 0$ for $j + 2i < 0$. Thus nonzero $a_{ij}$ occur only when $j + 2i \geqslant 0$. Here is a picture of the pairs $(i,j)$ satisfying this condition (as well as $j \geqslant 0$, $i \in \mathbb{Z}$):




(e) Consider a monomial $x^iy^j$ with $i \in \mathbb{Z}$, $j \geqslant 0$, and $j + 2i \geqslant 0$. If $j$ is even set $q = \frac{j}{2}$ and $p = i + \frac{j}{2}$. Then $q \geqslant 0$ since $j \geqslant 0$, and $p \geqslant 0$ since $2p = 2i + j \geqslant 0$. The monomial $x^py^q$ therefore belongs to $\mathcal{O}_X(X)$. Its restriction to $U_x$ is equal to

$$x^p\left(\frac{z^2}{x}\right)^q = x^{p-q}z^{2q} = x^{(i+\frac{j}{2})-\frac{j}{2}}z^{2(\frac{j}{2})} = x^iy^j.$$

On the other hand, if $j$ is odd, set $q = \frac{j-1}{2}$, $p = i + \frac{j-1}{2}$. Since $j \geqslant 0$ and is odd, $j \geqslant 1$ and therefore $q \geqslant 0$. Since $2i + j$ is $\geqslant 0$ and odd ($j$ is odd, and $2i$ even), $2i + j$ is $\geqslant 1$ and therefore $2i + j \geqslant 1$ and so $2p = 2i + (j-1) \geqslant 0$. The monomial $x^py^qz$ therefore belongs to $\mathcal{O}_X(X)$. Its restriction to $U_x$ is

$$x^p\left(\frac{z^2}{x}\right)^q z = x^{p-q}z^{2q+1} = x^{(i+\frac{j-1}{2})-\frac{j-1}{2}}z^{2(\frac{j-1}{2})+1} = x^iy^j.$$

(f) Given any $g_1$ and $g_2$ which agree on $U_{xy}$ as above, part (c) shows us that $g_1 = \sum a_{ij}x^iy^j$ with $i \in \mathbb{Z}$, $j \geqslant 0$, and $j + 2i \geqslant 0$. By part (e) any such monomial $x^iy^j$ is the restriction of a monomial of the form $x^py^qz^r$, with $p, q, r \geqslant 0$, is the restriction of something in $R[X] = \mathcal{O}_X(X)$. Thus the restriction map $R[X] = \mathcal{O}_X(X) \longrightarrow \mathcal{O}_X(U)$ is surjective.

3. Given a ring $A$ and an element $f \in A$ we have been looking at the ring $A[1/f]$ obtained by adjoining the additional element $1/f$ to $A$ (and of course using ring operations to get more elements). More precisely the ring $A[1/f]$ is the ring $A[y]/((1 - yf))$. There is a natural ring homomorphism $A \longrightarrow A[1/f]$, and we have seen in class that this is not always injective. For instance, if $h \in A$ is an element so that $h \cdot f^n = 0$ for some $n \geqslant 1$, then in $A[1/f]$ we compute that $h = (h \cdot f^n) \cdot \frac{1}{f^n} = 0 \cdot \frac{1}{f^n} = 0$.

The purpose of this question is to prove the converse direction: An element $h \in A$ is in the kernel of the map $A \longrightarrow A[1/f]$ only if there is an $n \geqslant 1$ such that $h \cdot f^n = 0$ in $A$.

Suppose that $h$ is such an element. This means that the image of $h$ under the inclusion $A \hookrightarrow A[y]$ must be in the ideal $(yf-1)$ in $A[y]$. Therefore there is a polynomial $g \in A[y]$ such that $h = g(yf-1)$. Since $g \in A[y]$ we can write $g$ as $g = g_0 + g_1 y + g_2 y^2 + \cdots + g_n y^n$ with each $g_j \in A$.

(a) Expand $g \cdot (yf-1)$ as a polynomial in $y$.

(b) As a polynomial in $y$, $h$ has degree 0. Since we have $h = g \cdot (yf-1)$, the coefficients of powers of $y$ on both sides of the equality must be the same. Comparing coefficients, write down all the relations you obtain.

(c) Show that $h \cdot f^{n+1} = 0$.

**Solution.**

(a) $$\begin{aligned} g(yf-1) &= (g_0 + g_1 y + g_2 y^2 + \cdots g_n y^n)(yf-1) \\ &= -g_0 + (fg_0 - g_1)y + (fg_1 - g_2)y^2 + \cdots + (fg_{n-1} - g_n)y^n + (fg_n)y^{n+1}. \end{aligned}$$

(b) Comparing powers of $y$ we have:

$$\begin{array}{rcl} h &=& -g_0, \\ 0 &=& fg_0 - g_1, \\ 0 &=& fg_1 - g_2, \\ \vdots & \vdots & \vdots \\ 0 &=& fg_{n-1} - g_n, \\ 0 &=& fg_n. \end{array}$$

(c) From the second through last equations we have $g_1 = fg_0$, $g_2 = fg_1 = f^2 g_0$, $g_3 = fg_2 = f^3 g_0$, $\ldots$, $g_n = fg_{n-1} = f^n g_0$, and finally $0 = fg_n = f^{n+1} g_0$. From the first equation we have $h = -g_0$, and therefore $f^{n+1} \cdot h = -f^n \cdot g_0 = 0$.

4. A topological space $X$ is called *quasi-compact* if whenever $\{U_i\}_{i \in S}$ are a family of open subsets such that $\cup_{i \in S} U_i = X$ then there are a finite number of $U_i$'s which actually cover $X$. (The term *compact* is reserved for Hausdorff topological spaces with this finite subcover property. A topological space with the finite subcover property alone is called quasi-compact.) In this question we will show that affine varieties are quasi-compact.

(a) Show that if $U_{f_i}$, $i \in S$ is a family of principal open subsets which cover an affine variety $X$, then there is a finite number which cover $X$. (Suggestion: Think about what the condition "the $U_{f_i}$ cover $X$" means about the complement.)

(b) Given an arbitrary cover of $X$ by open sets $\{U_j\}$, $j \in S$, use the fact that the principal open subsets are a basis for the Zariski topology and part (a) to show that a finite number of the $U_j$ are sufficient to cover $X$.

**Solution.**

(a) The condition that the "$U_{f_i}$ cover $X$" means that the complement $V(f_i)$ is empty, which is equivalent to the statement that the $f_i$ generate the ideal $R[X] = \mathcal{O}_x(X)$ (i.e. the whole ring). Since an ideal is equal to the whole ring if and only if 1 is in the ideal, this means that we must be able to write $1 = g_{i_1} f_{i_1} + g_{i_2} f_{i+2} + \cdots + g_{i_s} f_{i_s}$ for some finite number of $f_{i_1}, \ldots, f_{i_s}$. (The elements in an ideal generated by an infinite set are finite linear combinations of elements from the set.) Thus $(f_{i_1}, \ldots, f_{i_s}) = R[X]$ too, since 1 is in the ideal. But this means that $V(f_{i_1}, \ldots, f_{i_s}) = \varnothing$, or that $U_{f_{i_1}}, \ldots, U_{f_{i_s}}$ cover $X$.

Note: The finiteness argument above using the fact that 1 must be in the ideal generated by the $f_i$ may be replaced by a Notherian argument: We have a nested sequence of closed sets $V(f_1) \supseteq V(f_1, f_2) \supseteq V(f_1, f_2, f_3) \supseteq \cdots$ which can only have finitely many distinct elements (otherwise we would get an infinite sequence of distinct decreasing closed sets, which can't happen in a Notherian topology). Thus we only need finitely many of the $f_i$ to get down to the smallest closed set in the chain, the empty set.

(b) Suppose that $\{U_j\}$ is an arbitrary cover of $X$ by open sets. Since the principal open sets form a basis for the Zariski topology, we may cover each $U_j$ by principal open sets. This gives us a collection of principal open sets which also cover $X$ (since they cover the $U_j$, which cover $X$). By part (a), we only need finitely many of the principal open sets to cover $X$. For each principal open set in a finite subcover, take a $U_j$ which contains it. The union of these $U_j$ then contains $X$, since the principal open sets they contain cover $X$.

5. In class we have seen that if $X$ is an affine variety and $U_f$ a principal open subset, then $U_f$ is an affine variety. Perhaps every open subset is an affine variety? The purpose of this question is to show that the answer to this is no. Let $U = \mathbb{A}^2 \setminus \{(0,0)\}$. Recall that we have computed that $\mathcal{O}_{\mathbb{A}^2}(U) = k[x,y] = \mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2)$.

(a) Let $\varphi\colon U \hookrightarrow \mathbb{A}^2$ be the inclusion map. Compute the ring homomorphism $\varphi^*$.

(b) Maps between affine varieties are completely determined by the pullback maps. If $U$ were an affine variety, explain why $\varphi$ would have to be an isomorphism.

(c) Show that $U$ is not an affine variety.

**Solution.**

(a) The inclusion map $\varphi\colon U \longrightarrow \mathbb{A}^2$ is given by $\varphi(x_0, y_0) = (x_0, y_0) \in \mathbb{A}^2$. The map $\varphi^*$ therefore satisfies $\varphi^*(x) = x$ (since $\varphi^*(x)(x_0, y_0) = x(\varphi(x_0, y_0) = x_0$ for each $(x_0, y_0 \in \mathbb{A}^2)$. Similarly $\varphi^*(y) = y$. Thus the pullback map $\varphi^*$ is the identity map on

$$k[x, y] = \mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2) \xrightarrow{\varphi^*} \mathcal{O}_{\mathbb{A}^2}(U) = k[x, y].$$

(b) As part of proving the correspondence between maps between affine varieties and ring homomorphisms between their coordinate rings, we showed that $\varphi^*$ is an isomorphism if and only if $\varphi$ is an isomorphism. (See for instance Homework 3, question 1(d).)

(c) The map $\varphi\colon U \longrightarrow \mathbb{A}^2$ is not an isomorphism, since $\varphi$ is not surjective: $(0, 0)$ is not in the image. Therefore there can certainly not be any inverse map $\varphi^{-1}\colon \mathbb{A}^2 \longrightarrow U$, since there is nowhere to send $(0, 0)$.