1. Recall that an ideal $I \subseteq k[Z_0, \ldots, Z_n]$ is called a *homogeneous ideal* if, for every $f \in I$, when we write out f as a sum of homogeneous pieces, $f = F_0 + F_1 + F_2 + \cdots + F_d$, then each F_j is also in I.

In this problem we will show that an ideal I is homogeneous if and only if I can be generated by homogenous polynomials, i.e., if and only if there are homogeneous polynomials $G_1, \ldots, G_s \in k[Z_0, \ldots, Z_n]$ such that $I = \langle G_1, \ldots, G_s \rangle$.

(a) Assume that I is a homogeneous ideal. Show that I may be generated by homogeneous polynomials. (SUGGESTION: since I is an ideal in $k[Z_0, \ldots, Z_n]$ it may be generated by finitely many polynomials. Apply the 'homogeneous ideal' condition to these generators.)

(b) Suppose that $I = \langle G_1, \ldots, G_s \rangle$ with each G_i a homogenous polynomial. Show that I is a homogeneous ideal. (SUGGESTION: Any $f \in I$ can be written as $f = h_1 G_1 + \cdots + h_s G_s$ with the h_i polynomials in $Z_0, \ldots, Z_n$. Write each h_i as a sum of homogeneous pieces, expand the sum $\sum h_i G_i$, collect pieces of the same degree and compare with the homogeneous pieces of f.)

(c) Is the ideal $\langle X^3 - 5XZ^2 + Y^2 + XY,\ X^3 - 5XZ^2 - Y^2 - XY \rangle \subset \mathbb{Q}[X, Y, Z]$ homogeneous?

Solution.

(a) Since I is an ideal of $k[Z_0, \ldots, Z_n]$, by the Hilbert Basis Theorem there are finitely many polynomials $f_1, \ldots, f_r$ such that $I = (f_1, \ldots, f_r)$. Write each f_i as a sum of homogeneous pieces

$$f_i = F_{i,0} + F_{i,1} + \cdots + F_{i,j} + \cdots F_{i,d_i},$$

with each $F_{i,j}$ homogeneous of degree j.

By assumption, I is a homogeneous ideal, so each $F_{i,j} \in I$. The ideal generated by the $F_{i,j}$ is contained in I, since each F_{ij} is in I. However this ideal also contains the f_j, so must contain I. Thus $I = (\{F_{ij}\}_{i=1,j=0}^{i=r,j=d_i})$ is generated by finitely many homogeneous elements.

(b) Suppose that $I = (G_1, \ldots, G_s)$, with each G_j homogeneous of degree d_j. Suppose that $f \in I$, and write f as as sum of homogeneous pieces,

$$f = F_0 + F_1 + \cdots + F_d$$

with F_j homogeneous of degree j. Since $G_1,\ldots, G_s$ generate I, there are polynomials $h_1,\ldots, h_s \in k[Z_0,\ldots,Z_n]$ such that $f = h_1G_1 + h_2G_2 + \cdots + h_sG_s$. Write each h_j as a sum of homogeneous pieces

$$h_j = H_{j,0} + H_{j,1} + \cdots + H_{j,e_j}$$

with each $H_{j,i}$ homogeneous of degree i. Expanding $h_1G_1+\cdots+h_sG_s$ and collecting homogeneous pieces of the same degree, we see that the homogeneous piece of degree k of this sum is $\sum_{j=1}^{s} H_{j,k-d_j}G_j$. (Here $H_{k,k-d_j} = 0$ of $k - d_j$ is not in the range 0 to e_j.) Comparing with the homogeneous pieces of of f, we get the equality

$$F_k = \sum_{j=1}^{s} H_{j,k-d_j}G_j$$

for each $k = 0,\ldots, d$. Thus, each F_k is a combination of the generators of I and so is in I. Therefore, I is a homogeneous ideal.

(c) Yes, the ideal $I = \langle X^3 - 5XZ^2 + Y^2 + XY,\ X^3 - 5XZ^2 - Y^2 - XY\rangle \subset \mathbb{Q}[X,Y,Z]$ is a homogeneous ideal.

Write

$$f_1 = X^3 - 5XZ^2 + Y^2 + XY, \text{ and } f_2 = X^3 - 5XZ^2 - Y^2 - XY$$

set

$$G_1 = \tfrac{1}{2}(f_1 + f_2) = X^3 - 5XZ^2 \text{ and } G_2 = \tfrac{1}{2}(f_1 - f_2) = Y^2 + XY.$$

Since $G_1,\ G_2 \in \langle f_1, f_2\rangle$ we have $\langle G_1, G_2\rangle \subseteq I$. On the other hand, since $f_1 = G_1 + G_2$ and $f_2 = G_1 - G_2$, we also have the opposite inclusion $I \subseteq \langle G_1, G_2\rangle$, and therefore $I = \langle G_1, G_2\rangle$.

Since the ideal I can be generated by homogeneous polynomials, part (b) shows that I is a homogeneous ideal.

Note: The purpose of part (c) was to draw attention to the quantifiers in the characterization of homogeneous ideals : an ideal I is homogeneous if and only if there exists a set of homogeneous generators. This is not the same thing as saying that every set of generators of I must be homogeneous.

2. Recall that a subvariety $Y \subseteq \mathbb{A}^n$ is called a *cone* if whenever $p \in Y$ then $\lambda p \in Y$ for all $\lambda \in k^*$, where λp means the point obtained by scaling all the coordinates of p by λ. In this problem we will show that Y is a cone if and only if J_Y, the ideal of Y, is a homogeneous ideal. For this question we assume that k is an infinite field.

First suppose that Y is a cone, let f be an element of J_Y and write f as a sum of homogeneous pieces, $f = F_0 + F_1 + \cdots + F_d$, with each F_j homogeneous of degree j. To show that J_Y is a homogeneous ideal, we need to show that each $F_j \in J_Y$.

(a) Explain why it is sufficient to show that $F_j(p) = 0$ for all $p \in Y$.

(b) Fix $p \in Y$. By considering $f(\lambda p)$, explain why

$$0 = F_0(p) + \lambda F_1(p) + \lambda^2 F_2(p) + \cdots + \lambda^d F(p)$$

for all $\lambda \in k^*$.

(c) Considering the expression in (b) as a polynomial in λ, explain why we must have $F_j(p) = 0$ for each j, and hence (from the reductions above) that J_Y is a homogenous ideal.

(d) Now prove the other direction : assume that $Y \subseteq \mathbb{A}^n$ is a variety such that J_Y is a homogeneous ideal, and prove that Y is a cone. (The equivalence in question 1 may help.)

Solution.

(a) By definition of J_Y, a polynomial g is in J_Y if and only if $g(p) = 0$ for all $p \in Y$. Thus, showing that $F_j(p) = 0$ for all $p \in Y$ shows that $F_j \in J_Y$.

(b) Since Y is a cone, $\lambda p \in Y$ for all $\lambda \in k^*$, and so, since $f \in J_Y$, $f(\lambda p) = 0$ for all $\lambda \in k^*$. But by definition the F_j, $f(\lambda p) = F_0(\lambda p) + F_1(\lambda p) + \cdots + F_d(\lambda p)$. Since F_j is homogeneous of degree j, $F_j(\lambda p) = \lambda^j F_j(p)$. Putting these steps together we obtain

$$0 = f(\lambda p) = F_0(p) + \lambda F_1(p) + \lambda^2 F_2(p) + \cdots + \lambda^d F_d(p)$$

for all $\lambda \in k^*$.

(c) By part (b) the polynomial

$$F_0(p) + \lambda F_1(p) + \lambda^2 F_2(p) + \cdots + \lambda^d F_d(p)$$

is zero for all $\lambda \in k^*$. Since k is infinite, this means the polynomial has infinitely many roots, and that is only possible if the polynomial is the zero polynomial. Therefore its coefficients $F_0(p)$, $F_1(p)$, $\ldots$, $F_d(p)$ are zero. Since $p \in Y$ was arbitrary, this shows (by part (a)) that each $F_j \in J_Y$, and so J_Y is a homogeneous ideal.

(d) Conversely suppose that J_Y is homogeneous ideal. By **Q1(b)** this means we can assume that $J_Y = \langle G_1, G_2, \ldots, G_s \rangle$ for homogeneous polynomials $G_1, \ldots, G_s$, say of degrees d_1, d_2, $\ldots$, d_s.

We now show that Y is a cone. Suppose that $p \in Y$. By definition of J_Y this means that $G_1(p) = 0$, $G_2(p) = 0$, $\ldots$, $G_s(p) = 0$. For any $\lambda \in k^*$ we then have

$$G_1(\lambda p) = \lambda^{d_1} G_1(p) = \lambda^{d_1} \cdot 0 = 0,$$
$$G_2(\lambda p) = \lambda^{d_2} G_2(p) = \lambda^{d_2} \cdot 0 = 0,$$

all the way down to

$$G_s(\lambda p) = \lambda^{d_s} G_s(p) = \lambda^{d_s} \cdot 0 = 0.$$

We conclude that $\lambda p \in Y$, and so Y is a cone.

3. Suppose that U_0, U_1, and U_2 are the standard open subsets in $\mathbb{P}^2$, and that we have varieties $Y_0 \subset U_0$, $Y_1 \subset U_1$, and $Y_2 \subset U_2$, which agree on intersections. This means that $Y_i|_{U_i \cap U_j} = Y_j|_{U_i \cap U_j}$ for any i, j. In this question we will prove that there is a homogeneous ideal I in $k[X, Y, Z]$ so that if we set $Y = V(I)$ then $Y \cap U_i = Y_i$ for $i = 0, 1, 2$. I.e., we will show that if we define a subvariety of $\mathbb{P}^2$ as something obtained by glueing together affine varieties on the pieces, then this agrees with our definition of subvariety as something obtained by homogeneous polynomials.

Since Y_0, Y_1, and Y_2 are each affine varieties (in U_0, U_1, and U_2 respectively) each of them are given by ideals in their respective polynomial rings. Let I_0, I_1, and I_2 be these ideals. Then let $\widetilde{I}_0$, $\widetilde{I}_1$, and $\widetilde{I}_2$ be the homogenization of these ideals; namely the ideal obtained by homogenizing the polynomials in I_0, I_1, and I_2 respectively. These ideals have the property that $V(\widetilde{I}_j) \cap U_j = Y_j$ for each j. In other words, they each define projective varieties which restrict (separately) to the varieties we want on one of the open sets. However we do not know that $V(\widetilde{I}_j) \cap U_i = Y_i$ when $i \neq j$, so these ideals by themselves do not solve the problem.

Recall that $U_0 = \mathbb{P}^2 \setminus \{X = 0\}$, $U_1 = \mathbb{P}^2 \setminus \{Y = 0\}$, and $U_2 = \mathbb{P}^2 \setminus \{Z = 0\}$. Let $\widetilde{I}_0 X$ be the ideal obtained by multiplying all the elements of $\widetilde{I}_0$ by X. If we look at $V(\widetilde{I}_0 X)$, this variety contains all the points of $X = 0$ (i.e., all the points off of U_0), while we still have $V(\widetilde{I}_0 X) \cap U_0 = Y_0$. Similar statements hold for $V(\widetilde{I}_1 Y)$ and $V(\widetilde{I}_2 Z)$ (with similar definitions for $\widetilde{I}_1 Y$ and $\widetilde{I}_2 Z$).

Finally define $I = \widetilde{I}_0 X + \widetilde{I}_1 Y + \widetilde{I}_2 Z$, and set $Y = V(I)$. By our relation between subvarieties and geometric operations, this means that

$$Y = V(I) = V(\widetilde{I}_0 X) \cap V(\widetilde{I}_1 Y) \cap V(\widetilde{I}_2 Z).$$

We now want to show that $Y \cap U_i = Y_i$ for each i. By symmetry of the construction it is enough to do this for $i = 0$.

(a) Show that $Y \cap U_0 \subseteq Y_0$.

(b) Show that $Y_0 \subseteq Y \cap U_0$.

The proofs of both statements involve only elementary considerations about intersections, and inclusions, and the way that Y was defined. In particular, part (a) should be very straightforward. For part (b) you will need the condition that $Y_i|_{U_i \cap U_j} = Y_j|_{U_i \cap U_j}$.

Solution.

(a) From the condition that $V(\widetilde{I}_0 X) \cap U_0 = Y_0$, and the definition of Y we get

$$\begin{aligned} Y \cap U_0 &= V(\widetilde{I}_0 X) \cap V(\widetilde{I}_1 Y) \cap V(\widetilde{I}_2 Z) \cap U_0 \\ &= \Big(V(\widetilde{I}_0 X) \cap U_0\Big) \cap V(\widetilde{I}_1 Y) \cap V(\widetilde{I}_2 Z) \\ &= Y_0 \cap V(\widetilde{I}_1 Y) \cap V(\widetilde{I}_2 Z) \qquad \subseteq \; Y_0. \end{aligned}$$

(b) Conversely, suppose that $p \in Y_0$. Since Y is the intersection of $V(\widetilde{I}_0 X)$, $V(\widetilde{I}_1 Y)$, $V(\widetilde{I}_2 Z)$, to show that $p \in Y$, it suffices to show that p is in each of the varieties we are intersecting. Furthermore, since $p \in U_0$, it suffices to show that p is in each of $V(\widetilde{I}_0 X) \cap U_0$, $V(\widetilde{I}_1 Y) \cap U_0$, and $V(\widetilde{I}_2 Z) \cap U_0$. Let us consider each of these in turn.

- Since $p \in Y_0$, and since $V(\widetilde{I}_0 X) \cap U_0 = Y_0$, we certainly have $p \in V(\widetilde{I}_0 X) \cap U_0$.
- Consider two cases :
  - If p is on the line $Y = 0$, then $p \in V(\widetilde{I}_1 Y)$, since $V(\widetilde{I}_1 Y)$ contains the line $Y = 0$.
  - On the other hand, if p is not on the line $Y = 0$, then $p \in U_1$ and so $p \in U_0 \cap U_1$. From $p \in Y_0$ and the condition that $Y_0|_{U_0 \cap U_1} = Y_1|_{U_0 \cap U_1}$ we then conclude that $p \in Y_1$, and then from the condition that $V(\widetilde{I}_1 Y) \cap U_1 = Y_1$, that $p \in V(\widetilde{I}_1 Y)$.

  Thus, in either case, $p \in V(\widetilde{I}_1 Y)$.
- Similarly, for $V(\widetilde{I}_2 Z)$ we consider two cases.
  - If p is on the line $Z = 0$, then $p \in V(\widetilde{I}_2 Z)$, since $V(\widetilde{I}_2 Z)$ contains the line $Z = 0$.
  - On the other hand, if p is not on the line $Z = 0$, then $p \in U_2$ and so $p \in U_0 \cap U_2$. From $p \in Y_0$ and the condition that $Y_0|_{U_0 \cap U_2} = Y_2|_{U_0 \cap U_2}$ we then conclude that $p \in Y_2$, and then from the condition that $V(\widetilde{I}_2 Z) \cap U_2 = Y_2$, that $p \in V(\widetilde{I}_2 Z)$.

  Again we conclude that $p \in V(\widetilde{I}_2 Z)$.

Thus, $p \in V(\widetilde{I}_0 X) \cap V(\widetilde{I}_1 Y) \cap V(\widetilde{I}_2 Z) = Y$. Since $p \in Y_0$ was arbitrary, we conclude that $Y_0 \subseteq Y \cap U_0$.