1. Suppose that $C \subset \mathbb{P}^2$ is a curve, $q \in \mathbb{P}^2$ a point not on C, and ℓ a line not containing q. In class we saw how to use this setup to define a map $\varphi\colon C \longrightarrow \ell$. (The procedure was: for any $p \in C$, let $\overline{pq}$ be the line containing p and q, and define $\varphi(p)$ to be the intersection of C and ℓ.) In this question we will check that such a map is really a map of affine varieties.

We can make a useful simplification: We don't really need to think about C at all. Let $V = \mathbb{P}^2 \setminus \{q\}$. Then the procedure above really defines a map $\psi\colon V \longrightarrow \ell$. The map φ is the composite of ψ with the inclusion $C \hookrightarrow V$. Since inclusion is an algebraic map, and compositions of algebraic maps are algebraic maps, all we really need to do is to verify that ψ is an algebraic map.

Let $q = [0 : 0 : 1]$ and ℓ be the line $Z = 0$.

(a) Let $p = [\alpha\colon \beta\colon \gamma]$ be a point of V. Write down the equation of the unique line in $\mathbb{P}^2$ which contains p and q.

(b) Compute the intersection of the line above with ℓ (i.e., calculate $\psi(p)$).

From your answer in (b), it will be clear that projection from q looks like an algebraic map. However, let's practice computing in coordinates by examining this map in coordinate charts. The open set V is covered by the standard coordinate charts U_0 and U_1.

(c) Explain what the line ℓ looks like in the coordinate system of U_0, and then write down the formula for the map $U_0 \longrightarrow (\ell \cap U_0)$ given by restricting ψ to U_0. (I.e, if $p = (y_0, z_0)$ is a point of U_0, what point on $(U_0 \cap \ell)$ is $\psi(p)$?)

(d) Do the same thing for U_1.

Now let see what φ looks like near $p \in C$. We will also have to deal with an issue not raised in class : when $q = p$, what does the "line containing p and q" mean? By taking the limit as $q \to p$, you may be convinced that this should mean : use the tangent line to C at p (and that *is* what it should mean).

Rather than do the general case, we will pick a specific curve and see that the construction works there. Let C be the conic given by $YZ - X^2 = 0$ (and p and L as above). Let us look at C in the remaining chart, U_2. In this chart p becomes the point $(0, 0)$, while the line L, given by $Z = 0$, does not appear in U_2.

(e) Dehomogenize the equation of C on the chart U_2 (i.e., with respect to Z).

(*f*) Suppose that $q \in C \cap U_2$, $q \neq p$, writing $q = (x, y)$ $(=[x : y : 1])$, where does your formula from (a) say that $\overline{pq}$ intersects the "line at infinity" L?

(*g*) Find the limit of your answer in (*f*) as $q \to p$. You will probably have to use the fact that the x and y coordinates of q satisfy the equation you found in (e).

(*h*) Find the tangent line to C at p in the chart U_2, and verify that the intersection of the tangent line and L is the same as your answer from (g).

Solution.

(a) $[\alpha : \beta : \gamma] \neq [0 : 0 : 1]$ (by hypothesis), at least one of α and β is nonzero. The equation $\beta X - \alpha Y = 0$ is therefore nonzero and defines a line. By plugging in to the equation, we see that the line contains the points $[\alpha : \beta : \gamma]$ and the point $[0 : 0 : 1]$. Thus this is the unique line containing these two points.

(b) The equations $\beta X - \alpha Y = 0$ and $Z = 0$ are not scalar multiples of each other and so define distinct lines. The point $[\alpha : \beta : 0]$ satisfies both equations and so must be the unique point of intersection of the two lines. Thus, for a point $[\alpha : \beta : \gamma] \in V$ we have $\psi([\alpha : \beta : \gamma]) = [\alpha : \beta : 0] \in \ell$. In other words, in these coordinates, the map ψ is simply "set the last coordinate to 0".

(c) On U_0, where $X \neq 0$, the coordinates are $y_0 = \frac{Y}{X}$ and $z_0 = \frac{Z}{X}$. The line $Z = 0$ in $\mathbb{P}^2$ is the line $z_0 = 0$ in U_0. The point $[\alpha : \beta : \gamma]$ is (if $\alpha \neq 0$) the point $\left(\frac{\beta}{\alpha}, \frac{\gamma}{\alpha}\right)$ in U_0. The map ψ sends this point to $[\alpha : \beta : 0]$, which is the point $\left(\frac{\beta}{\alpha}, 0\right)$ in U_0. In other words, on U_0 the map ψ looks like

$$\psi(y_0, z_0) = (y_0, 0).$$

(d) On U_1, where $Y \neq 0$, the coordinates are $x_1 = \frac{X}{Y}$ and $z_1 = \frac{X}{Y}$. The line $Z = 0$ in $\mathbb{P}^2$ is the line $z_1 = 0$ in U_1. The point $[\alpha : \beta : \gamma]$ is (if $\beta \neq 0$) the point $\left(\frac{\alpha}{\beta}, \frac{\gamma}{\beta}\right)$ in U_1. The map ψ sends this point to $[\alpha : \beta : 0]$, which is the point $\left(\frac{\alpha}{\beta}, 0\right)$ in U_1. In other words, on U_1 the map ψ looks like

$$\psi(x_1, z_1) = (x_1, 0).$$

(e) Dehomogenizing $YZ - X^2$ with respect to Z we get $y - x^2$, so that $y - x^2 = 0$ is the parabola $y = x^2$.

(*f*) If $q = (x, y) = [x : y : 1]$, then the formula from (a) says that $\overline{pq}$ intersects L at $[x : y : 0]$.

(g) If q is also on C, then $y = x^2$, so q is of the form (x, x^2) and $\varphi(q) = [x : x^2 : 0] = [1 : x : 0]$. The point p is the point $(0,0)$ in U_2. Taking the limit of $\varphi(q)$ as $q \to p$ is taking the limit as $x \to 0$ of $[1 : x : 0]$, and the limit is clearly $\lim_{x\to 0}[1 : x : 0] = [1 : 0 : 0]$.

(h) The tangent line to $y = x^2$ at $(0,0)$ is the x-axis, namely $y = 0$. Homogonizing, this is the line $Y = 0$, and intersects the line L (which is $Z = 0$), in the point $[1 : 0 : 0]$, just as we found in (g).

These maps are "projection onto $Z = 0$" in there respective coordinates, and are certainly algebraic maps.

2. Singular points and the topology of a curve.

(a) Find the unique singular point of the curve $6Y^2Z^2 = 6X^2Z^2 - 8X^3Z + 4Y^3Z + 3X^4$ in $\mathbb{P}^2$. Look at the equation in an affine chart of the singular point, and show that analytically it is a *node*.

(b) Draw a "balloon picture" of the topological shape of this curve. (Hints: The curve has degree 4, so you know what it looks like when it is smoothed. The curve is also irreducible, so only has one piece.)

Solution.

(a) Let $F = 6Y^2Z^2 - 6X^2Z^2 + 8X^3Z - 4Y^3Z - 3X^4$. Then the conditions are:

F_X	$=$	$-12XZ^2 + 24X^2Z - 12X^3$	$=$	0	$\Longrightarrow$	$-12X(X-Z)^2 = 0$ so $X = 0$ or $X = Z$
F_Y	$=$	$12YZ^2$	$=$	0	$\Longrightarrow$	$Y = 0$ or $Z = 0$
F_Z	$=$	$12Y^2Z - 12X^2Z + 8X^3 - 4Y^3$	$=$	0		

We consider several cases, starting with the condition imposed by $F_Y = 0$, that either $Y = 0$ or $Z = 0$.

$\boxed{Z = 0}$: Then the F_X condition implies that either $X = 0$ or $X = Z = 0$, so in either case $X = 0$. But then the F_Z condition becomes $-4Y^3 = 0$ so that $Y = 0$. Since $[0 : 0 : 0]$ is not a point of $\mathbb{P}^2$ we conclude that there are no solutions with $Z = 0$.

$\boxed{Y = 0}$: Then the F_X condition implies that either $X = 0$ or $X = Z$. Let us deal with each of these subcases in turn.

- $\boxed{X=0}$: Substituting $X=0$ and $Y=0$ into F_Z results in 0. Thus, when $X=Y=0$, $F_X=F_Y=F_Z=0$, and therefore the point $[0:0:1]$ is a singular point of $F=0$.
- $\boxed{X=Z}$: Substituting $Y=0$ and $X=Z$ in to F_Z gives $-4Z^3$, so the condition $F_Z=0$ implies that $Z=0$. But then because $X=Z$ we have $X=0$ too. However $X=Y=Z=0$ is not a point of $\mathbb{P}^2$, so this is not a solution.

Thus the unique singular point of $F=0$ is $p=[0:0:1]$. Dehomogenizing with respect to Z, F becomes

$$f=6(y^2-x^2)+(8x^3-4y^3)+3x^4,$$

(the grouping above is by degree of homogenous piece). In this chart the singular point is $(0,0)$. The polynomial f has no linear term and so is certainly singular at $(0,0)$. The degree two homogeneous piece factors as $6(x+y)(x-y)$. Since the factorization has two distinct linear factors, the singularity must be a node, as was mentioned in class. We now check this by making changes of variables.

First, regroup the monomials in f, collecting powers of x and y separately.

$$f=y^2(6-4y)-x^2(6-8x-3x^2).$$

Since $6-4y$ is not zero at $(0,0)$, this function has a square root near $(0,0)$. Similarly, $6-8x-3x^2$ has a square root near $(0,0)$. Let $h_1=\sqrt{6-4y}$ and $h_2=\sqrt{6-8x-3x^2}$ be these square roots (they are well determined up to sign). Using the formula

$$\sqrt{1-t}=1-\tfrac{1}{2}t-\tfrac{1}{8}t^2-\tfrac{1}{16}t^3-\tfrac{5}{128}t^4-\cdots-\tfrac{1\cdot3\cdot5\cdots(2n-3)}{2^n\cdot n!}t^n-\cdots$$

from first year calculus, we compute that the power series expansions of h_1 and h_2 near $(0,0)$ are

$$\begin{aligned} h_1(x,y) &= \sqrt{6}\left(1-\tfrac{1}{3}y-\tfrac{1}{18}y^2-\tfrac{1}{54}y^3-\cdots\right)\\ h_2(x,y) &= \sqrt{6}\left(1-\tfrac{2}{3}x-\tfrac{17}{36}x^2-\tfrac{17}{54}x^3-\cdots\right). \end{aligned}$$

Set $x_1=x\cdot h_2(x,y)$ and $y_1=y\cdot h_1(x,y)$. Then the determinant of the Jacobian matrix of the functions x_1, y_1 at $(0,0)$ is

$$\begin{array}{cc} & \begin{array}{cc} x_1 & y_1 \end{array} \\ \begin{array}{c} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{array} & \begin{vmatrix} \sqrt{6} & 0 \\ 0 & \sqrt{6} \end{vmatrix} \end{array} = 6 \neq 0,$$

and so x_1 and y_1 are valid coordinate functions near $(0,0)$. Changing coordinates, the function f becomes

$$f = y_1^2 - x_1^2.$$

Set $x_2 = y_1 + x_1$ and $y_2 = y_1 - x_1$. The Jacobian matrix of these functions at $(0,0)$ is

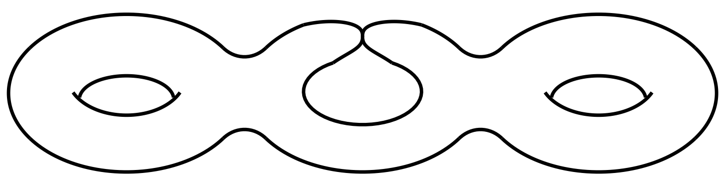
$$\begin{array}{cc} & \begin{matrix} x_2 & y_2 \end{matrix} \\ \begin{matrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial y_1} \end{matrix} & \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} \end{array} = 2 \neq 0,$$

and so again x_2 and y_2 are valid coordinate funtions near $(0,0)$. In these new coordinates f becomes

$$f = x_2 y_2,$$

so we see that $f = 0$ does indeed have a node at $(0,0)$.

(b) We know that C is irreducible (in this case this simply means that F doesn't factor). If we smooth C we will obtain a curve of genus $\binom{4-1}{2} = 3$, and thus C must look something like this:



3. In class we figured out the genus of a degree d plane curve by taking the union of a degree $(d-1)$ curve and a line, and smoothing it. When seeing a new type of argument, it is good to check for *consistency*: If the argument is applied (correctly) in a similar way, it should also lead to correct conclusions. For instance, instead of smoothing a degree $(d-1)$ curve and a degree 1 curve, why not take the union of d lines and smooth them?

(a) Suppose that $C_1, \ldots, C_r$ are curves in $\mathbb{P}^2$ of degrees $d_1, \ldots, d_r$. Show that their union is a curve of degree $d_1 + d_2 + \cdots + d_r$. (Suggestion: What is the definition of a "curve of degree d"?).

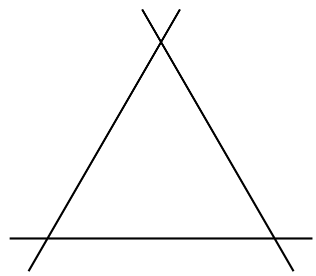
(b) Now let C be the union of three distinct lines. By part (a) C is a (singular) curve of degree 3. Draw a the real picture of an intersection of three lines. How many nodes does C have? Draw the "balloon picture" of the nodal curve C, and then explain which genus Riemann surface is obtained when the curve is smoothed. Does this agree with our formula?

(c) Do the same thing for the union of four distinct lines in $\mathbb{P}^2$. You should suppose that the lines are general enough so that all the singularities are nodes. (For instance, while any pair of lines must intersect, three lines should never all meet in a single point.)

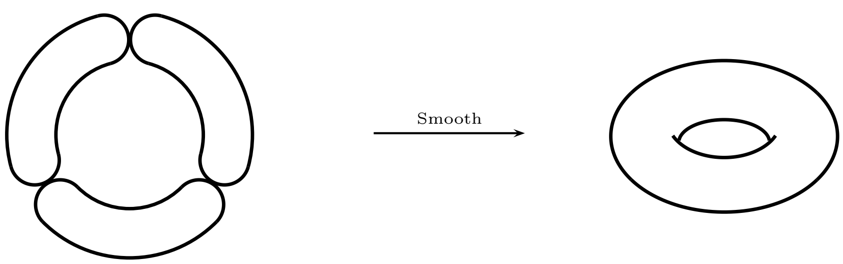
Solution.

(a) By definition a curve of degree d in $\mathbb{P}^2$ is given by $F = 0$, where F is a homogeneous equation of degree d in X, Y, and Z. Suppose that $C_1,\dots, C_r$ are curves of degrees given as $F_1 = 0$, $F_2 = 0$, $\dots$, $F_r = 0$, where each F_j is of degree d_j. Then their union is the set of zeros of $F = F_1F_2\cdots F_r$, which is homogeneous of degree $d_1 + d_2 + \cdots + d_r$.

(b) Three general lines will intersect like this:

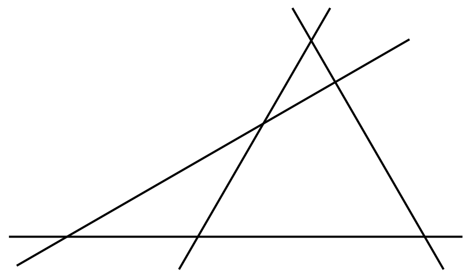


From the picture, the union of three lines has 3 nodes. The corresponding "balloon picture" and its smoothing is this:

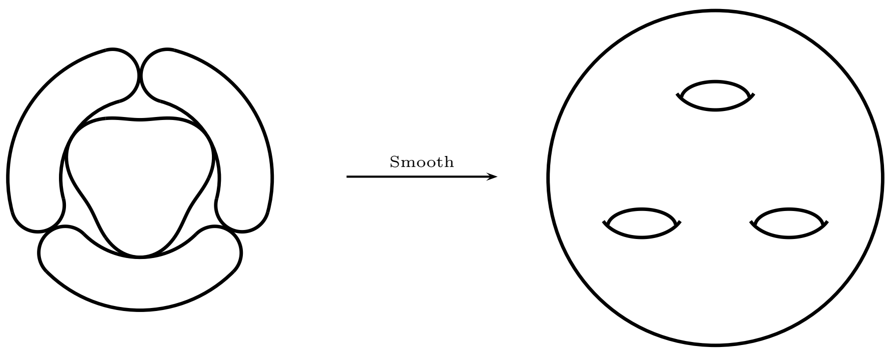


This agrees with our previous argument: a smooth curve in $\mathbb{P}^2$ of degree 3 should have genus 1.

(c) Four general lines in $\mathbb{P}^2$ intersect in a picture something like this:



Each pair of lines intersects in a unique point, so there are a total of $\binom{4}{2} = 6$ nodes. A "balloon picture" and its smoothing is:



This also agrees with our previous argument. A smooth curve of degree 4 in $\mathbb{P}^2$ should have genus 3.