DUE DATE: JAN. 15, 2019

1. Suppose that $X \subset \mathbb{R}^n$ is a shape.

(a) If f_1 and f_2 are functions on $\mathbb{R}^n$, show that $f_1 = f_2$ on X (i.e., when restricted to X) if and only if $f_1 - f_2$ is zero on X.

(b) If g is a function on $\mathbb{R}^n$ which is zero when restricted to X, and h any function on $\mathbb{R}^n$, show that hg is zero when restricted to X.

(c) Now let X be the circle $\{(x,y) \mid x^2+y^2=1\} \subset \mathbb{R}^2$. Take the following functions on $\mathbb{R}^2$ and organize them into groups according to their equality when restricted to X:

(**1**) 1; (**2**) y; (**3**) x^2+y^2; (**4**) x^2-y^2;

(**5**) $2x^2+1$; (**6**) $2x^2-1$; (**7**) x^4-y^4; (**8**) y^3+x^2y.

(I.e, group together the functions which are equal when restricted to X.)

[Math 813 only] (d) Let X be the unit circle as in part (c). Let $f(x,y)$ be any polynomial in x and y. Prove that there is a polynomial of the form $g(x,y) = g_0(x) + g_1(x)y$ such that the restriction of f to X is equal to the restriction of g to X.

2. Let X be the unit circle $\{(x,y) \mid x^2+y^2=1\} \subset \mathbb{R}^2$ and Y the unit sphere $\{(u,v,w) \mid u^2+v^2+w^2=1\} \subset \mathbb{R}^3$. Define a map $\varphi: X \longrightarrow Y$ by the rule $\varphi(x,y) = (xy, y^2, x)$.

(a) Show that φ is well-defined. That is, show that if $(x,y) \in X$ then $\varphi(x,y) \in Y$.

(b) Compute $\varphi^*(u)$, $\varphi^*(v)$, and $\varphi^*(w)$.

(c) Compute $\varphi^*(3u^2 - 2vw + 5)$.

(d) Let f be the function $5xy^3+7x^2-9y^2$ restricted to X. Find a polynomial $g(u,v,w)$ on $\mathbb{R}^3$ so that $f = \varphi^*(g)$.

3. Let $X = \mathbb{R}$ and $Y = \mathbb{R}^2$. The ring of polynomial functions on X is $\mathbb{R}[x]$. The ring of polynomial functions on Y is $\mathbb{R}[x,y]$.

(a) The ring $\mathbb{R}[x]$ is a subring of $\mathbb{R}[x,y]$, i.e., the inclusion map $\psi_1: \mathbb{R}[x] \longrightarrow \mathbb{R}[x,y]$ is a ring homomorphism. Find a map $\varphi_1: Y \longrightarrow X$ such that pullback by φ_1 induces ψ_1. (I.e., "$\varphi_1^* = \psi_1$".)

(b) The map $\psi_2: \mathbb{R}[x,y] \longrightarrow \mathbb{R}[x]$ given by "setting $y = 0$" (i.e., $\psi_2(f(x,y) = f(x,0))$ is also a ring homomorphism. Find a map $\varphi_2: X \longrightarrow Y$ so that $\varphi_2^* = \psi_2$.

(c) How would you describe these maps geometrically? (I.e., in a picture or in words, what do they do?)

MINOR SUGGESTION: The fact that there is more than one x may make things more confusing. Relabelling one set of variables and describing the ring homomorphisms in the new variables may make things a bit clearer.