Math 413/813 — Homework Assignment 3

DUE DATE: JAN. 29, 2019

1. Let X and Y be two affine varieties, with rings of functions $R[X]$ and $R[Y]$. In this problem we will use the theorem from the classes of Jan. 17th and 21st to prove that X and Y are isomorphic varieties if and only if $R[X]$ and $R[Y]$ are isomorphic rings.

(a) Explain why $(1_X)^* = 1_{R[X]}$

Here 1_X and $1_{R[X]}$ are being used in the category-theoretic sense. They are, respectively, the identity morphism $1_X\colon X \longrightarrow X$ and the identity ring homomorphism $1_{R[X]}\colon R[X] \longrightarrow R[X]$.

(b) Suppose that $\varphi\colon X \longrightarrow X$ is a morphism of affine varieties and that $\varphi^* = 1_{R[X]}$. Explain why must have $\varphi = 1_X$.

(c) Suppose that X and Y are isomorphic affine varieties. Writing out the definition of "isomorphic varities" and applying the functor to rings, explain why $R[X]$ and $R[Y]$ are isomorphic rings.

(d) Now suppose that $R[X]$ and $R[Y]$ are isomorphic rings. Write out the definition of "isomorphic rings" and use part (c) of the theorem as well as (b) above to show that X and Y are isomorphic varieties.

2. In this question we will see an example of a morphism of affine varieties which is a bijection on points, but which is not an isomorphism. (In other words, in the category of affine varieties, isomorphism implies more than just bijection.) Let $X = \mathbb{A}^1$ with ring of functions $k[t]$, and let Y be the subset of $\mathbb{A}^2$ given by the equation $y^2 = x^3$.

(a) Let $\varphi\colon X \longrightarrow \mathbb{A}^2$ be the map given by $\varphi(t) = (t^2, t^3)$. Show the image of φ lies in Y, so that φ defines a morphism $\varphi\colon X \longrightarrow Y$.

(b) Show that φ is surjective. (i.e., given $(x, y) \in Y$, show that there is a t such that $\varphi(t) = (x, y)$.)

(c) Show that φ is injective.

(d) Draw a sketch of Y ($\mathbb{R}^2$ points only). One suggestion: from part (b) you know that Y is the image of φ, so you can use the parameterization given by φ to see what Y looks like.

(e) Compute the image of the ring homomorphism $\varphi^*\colon R[Y] \longrightarrow R[X]$ (and recall that $R[X] = k[t]$). Is φ^* surjective?

(f) Explain why φ is not an isomorphism of affine varieties.

3. Consider the following four affine varieties, all contained in $\mathbb{A}^3$.

$$\begin{aligned} X &= \left\{(x_1,x_2,x_3) \,\middle|\, x_1^2+x_2^2-1=0\right\} \subset \mathbb{A}^3 \\ Y &= \left\{(y_1,y_2,y_3) \,\middle|\, y_1^2+y_2^2-y_3^2=0\right\} \subset \mathbb{A}^3 \\ Z &= \left\{(z_1,z_2,z_3) \,\middle|\, z_1^2+z_2^2+z_3^2-625=0\right\} \subset \mathbb{A}^3 \\ W &= \left\{(w_1,w_2,w_3) \,\middle|\, w_1^2+w_2^2-w_3=0\right\} \subset \mathbb{A}^3 \end{aligned}$$

(a) Draw sketches of X, Y, Z, and W.

Define a map $\varphi_1 \colon X \longrightarrow \mathbb{A}^3$ by $\varphi_1(x_1,x_2,x_3) = (x_1x_3, x_2x_3, x_3)$.

(b) Is the image of φ_1 contained in Y, Z, or W? (Justify your answer.)

Define a map $\varphi_2 \colon X \longrightarrow \mathbb{A}^3$ by $\varphi_2(x_1,x_2,x_3) = (-9x_1+12x_2, 12x_1-16x_2, 20x_1+15x_2)$.

(c) Is the image of φ_2 contained in Y, Z, or W? (Justify your answer.)

Define a map $\varphi_3 \colon Y \longrightarrow \mathbb{A}^3$ by $\varphi_3(y_1,y_2,y_3) = (y_1,y_2,y_3^2)$.

(d) Is the image of φ_3 contained in X, Z, or W? (Justify your answer.)

One of the maps (b)–(d) has image in W.

(e) What is the pullback of $3\overline{w}_1 - \overline{w}_2^2 + \overline{w}_3 \in R[W]$ under this map?

Now we will try and go the other way, from a map of rings to a map of varieties. Define a ring homomorphism

$$R[X] = \frac{k[x_1,x_2,x_3]}{\langle x_1^2+x_2^2-1\rangle} \longleftarrow \frac{k[w_1,w_2,w_3]}{\langle w_1^2+w_2^2-w_3\rangle} = R[W] \colon \psi$$

by the rule $\psi(\overline{w}_1) = 2\overline{x}_1$, $\psi(\overline{w}_2) = 2\overline{x}_2$, $\psi(\overline{w}_3) = 4$.

(f) Check that this ring homomorphism is well-defined by showing that $\psi(\overline{w}_1^2 + \overline{w}_2^2 - \overline{w}_3) = 0$.

(g) What geometric map $\varphi \colon X \longrightarrow W$ does the ring homomorphism ψ correspond to? (Write your formula for φ in the form $\varphi(x_1,x_2,x_3) =$ (formulas in x_1, x_2, x_3) $\subset \mathbb{A}^3$ as in (b)–(d) above.)