1. In this problem we will use parts of the algebra-geometry correspondence that we have built up to prove the following result in commutative algebra:

> Let $I \subseteq k[x_1, \dots, x_n]$ be an ideal, and $\overline{I}$ the intersection of all the maximal ideals of $k[x_1, \dots, x_n]$ containing I. Then $\overline{I} = \sqrt{I}$.

(a) Show that a maximal ideal is a radical ideal. (SUGGESTION: It may help to rewrite the condition that $I \subset A$ is a radical ideal in terms of the quotient ring A/I.)

(b) Show that an arbitrary intersection of maximal ideals is a radical ideal. (You can use results from the previous homework.)

Now assume that $J \subseteq k[x_1, \dots, x_n]$ is a radical ideal, and let $\overline{I}$ be the intersection of all maximal ideals containing J. I.e., $\overline{I} = \bigcap_{J \subseteq \mathfrak{m}} \mathfrak{m}$, where each $\mathfrak{m}$ is a maximal ideal.

(c) Show that every maximal ideal containing J also contains $\overline{I}$.

(d) Show that $J \subseteq \overline{I}$.

(e) Show that every maximal ideal containing $\overline{I}$ also contains J.

Next, using parts of the algebra-geometry dictionary we have seen in class:

(f) Explain why $V(J) = V(\overline{I})$. (SUGGESTION: what do (c) and (e) say about the points of $V(J)$ and $V(\overline{I})$?)

(g) Explain why we then know that $J = \overline{I}$.

Finally, let $I \subset k[x_1, \dots, x_n]$ be any ideal, and set $J = \sqrt{I}$.

(h) Show that any maximal ideal containing I also contains J.

(i) Show that any maximal ideal containing J also contains I.

(j) Prove the commutative algebra statement above.

2. In this question we will explore the construction of sum of ideals. Given a ring A, and a (possibly infinite) collection of ideals $I_\alpha \subset A$, $\alpha \in S$ recall that we have defined $\sum_{\alpha \in S} I_\alpha$ as all possible finite sums of elements in the I_α, i.e.,

$$\sum_{\alpha \in S} I_\alpha = \left\{ f_{\alpha_1} + f_{\alpha_2} + \cdots + f_{\alpha_k} \;\middle|\; f_{\alpha_j} \in I_{\alpha_j} \right\}.$$

(a) Show that $\sum_{\alpha \in S} I_\alpha$ is an ideal.

(b) Suppose that A is a Noetherian ring. Show that there is a finite subset $S' \subseteq S$ such that $\sum_{\alpha \in S'} I_\alpha = \sum_{\alpha \in S} I_\alpha$.

(c) Suppose that X is an affine variety with ring of functions $R[X]$. Let Z_α, $\alpha \in S$ be a collection of closed subsets of X corresponding to ideals J_α, $\alpha \in S$. Show that

$$V(\sum_{\alpha \in S} J_\alpha) = \bigcap_{\alpha \in S} Z_\alpha$$

as claimed in class.

3. The *elementary symmetric polynomials* in x_1, x_2, and x_3 are the polynomials $e_1 = x_1 + x_2 + x_3$, $e_2 = x_1x_2 + x_2x_3 + x_1x_3$, and $e_3 = x_1x_2x_3$. It is a useful result in algebra that these polynomials are algebraically independent over any field. This means that for any polynomial $f(y_1, y_2, y_3) \in k[y_1, y_2, y_3]$ the polynomial $f(e_1, e_2, e_3) \in k[x_1, x_2, x_3]$ is zero only if f was zero to start with.

In contrast, the functions $g_1 = x_1^2$, $g_2 = x_1x_2$, and $g_3 = x_2^2$ are not algebraically independent. Letting $f(y_1, y_2, y_3) = y_1y_3 - y_2^2$, we have $f \neq 0$ but $f(g_1, g_2, g_3) = 0$.

In this problem we will use combination of geometric and algebraic arguments (and thus the algebra $\leftrightarrow$ geometry dictionary) to show that e_1, e_2, and e_3 are algebraically independent.

(a) Suppose that $\varphi \colon X \longrightarrow Y$ is a morphism of affine varieties, and that φ is surjective. Show that the homomorphism $\varphi^* \colon R[Y] \longrightarrow R[X]$ is injective.

(b) Let $X = \mathbb{A}^3$ with ring of functions $k[x_1, x_2, x_3]$, and let Y also be $\mathbb{A}^3$ with ring of functions $k[y_1, y_2, y_3]$. Let $\varphi \colon X \longrightarrow Y$ be the map

$$\varphi(x_1, x_2, x_3) = \Big(x_1 + x_2 + x_3,\ x_1x_2 + x_2x_3 + x_1x_3,\ x_1x_2x_3\Big).$$

So, for instance, $\varphi(3,\ 1,\ 5) = (3 + 1 + 5,\ 3 \cdot 1 + 1 \cdot 5 + 3 \cdot 5,\ 3 \cdot 1 \cdot 5) = (9,\ 23,\ 15)$.

Describe the pullback map φ^*. In particular, what are $\varphi^*(y_1)$, $\varphi^*(y_2)$, and $\varphi^*(y_3)$?

(c) Expand the product $(t - \alpha)(t - \beta)(t - \gamma)$.

(d) For any $(a, b, c) \in Y$, consider the polynomial $t^3 - at^2 + bt - c$ and let α, β, and γ be the roots. Show that $\varphi(\alpha, \beta, \gamma) = (a, b, c)$.

(e) Prove that e_1, e_2, and e_3 are algebraically independent.