Math 413/813 — Homework Assignment 10

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1. Suppose that $C \subset \mathbb{P}^2$ is a curve, $p \in C$ and L a line not containing p. In class we saw how to use this setup to define a map $\varphi\colon C \longrightarrow L$. (The procedure was: for any $q \in C$, let $\overline{pq}$ be the line containing p and q, and define $\varphi(q)$ to be the intersection of $\overline{pq}$ and L.) In this question we will check that such a map is really a map of varieties.

We can make a useful simplification: We mostly don't need to think about C at all. Let $V = \mathbb{P}^2 \setminus \{p\}$. Then the procedure above really defines a map $\psi\colon V \longrightarrow L$. Let $C_0 = C \setminus \{p\}$. Restricted to C_0, the map φ is the composite of ψ with the inclusion $C_0 \hookrightarrow V$. Since inclusion is an algebraic map, and compositions of algebraic maps are algebraic maps, if we verify that ψ is an algebraic map we will have shown that $\varphi|_{C_0}$ is an algebraic map.

Let $p = [0:0:1]$ and L be the line $Z = 0$.

(*a*) Let $q = [\alpha\colon \beta\colon \gamma]$ be a point of V. Write down the equation of the unique line in $\mathbb{P}^2$ which contains p and q.

(*b*) Compute the intersection of the line above with L (i.e., calculate $\psi(q)$).

From your answer in (*b*), it will be clear that projection from p looks like an algebraic map (i.e., "given by algebraic formulae"). However, let's practice computing in coordinates by examining this map in coordinate charts. The open set V is covered by the standard coordinate charts U_0 and U_1.

(*c*) Explain what the line L looks like in the coordinate system of U_0, and then write down the formula for the map $U_0 \longrightarrow (L \cap U_0)$ given by restricting ψ to U_0. (I.e, if $q = (y_0, z_0)$ is a point of U_0, what point on $(U_0 \cap L)$ is $\psi(q)$?)

(*d*) Do the same thing for U_1.

Now let see what φ looks like near $p \in C$. We will also have to deal with an issue not raised in class : when $q = p$, what does the "line containing p and q" mean? By taking the limit as $q \to p$, you may be convinced that this should mean : use the tangent line to C at p (and that *is* what it should mean).

Rather than do the general case, we will pick a specific curve and see that the construction works there. Let C be the conic given by $YZ - X^2 = 0$ (and p and L as above). Let us look at C in the remaining chart, U_2. In this chart p becomes the point $(0, 0)$, while the line L, given by $Z = 0$, does not appear in U_2.

(*e*) Dehomogenize the equation of C on the chart U_2 (i.e., with respect to Z).

(*f*) Suppose that $q \in C \cap U_2$, $q \neq p$, writing $q = (x, y)$ (=$[x : y : 1]$), where does your formula from (*a*) say that $\overline{pq}$ intersects the "line at infinity" L?

(*g*) Find the limit of your answer in (*f*) as $q \to p$. You will probably have to use the fact that the x and y coordinates of q satisfy the equation you found in (*e*).

(*h*) Find the tangent line to C at p in the chart U_2, and verify that the intersection of the tangent line and L is the same as your answer from (*g*).

2. Singular points and the topology of a curve.

(*a*) Find the unique singular point of the curve $6Y^2Z^2 = 6X^2Z^2 - 8X^3Z + 4Y^3Z + 3X^4$ in $\mathbb{P}^2$. Look at the equation in an affine chart of the singular point, and show that analytically it is a *node*.

(*b*) Draw a "balloon picture" of the topological shape of this curve. (Hints: The curve has degree 4, so you know what it looks like when it is smoothed. The curve is also irreducible, so only has one piece.)

3. In class we figured out the genus of a degree d plane curve by taking the union of a degree $(d-1)$ curve and a line, and smoothing it. When seeing a new type of argument, it is good to check for *consistency*: If the argument is applied (correctly) in a similar way, it should also lead to correct conclusions. For instance, instead of smoothing a degree $(d-1)$ curve and a degree 1 curve, why not take the union of d lines and smooth them?

(*a*) Suppose that $C_1, \ldots, C_r$ are curves in $\mathbb{P}^2$ of degrees $d_1, \ldots, d_r$. Show that their union is a curve of degree $d_1 + d_2 + \cdots + d_r$. (Suggestion: What is the definition of a "curve of degree d"?).

(*b*) Now let C be the union of three distinct lines. By part (*a*) C is a (singular) curve of degree 3. Draw a the real picture of an intersection of three lines. How many nodes does C have? Draw the "balloon picture" of the nodal curve C, and then explain which genus Riemann surface is obtained when the curve is smoothed. Does this agree with our formula?

(*c*) Do the same thing for the union of four distinct lines in $\mathbb{P}^2$. You should suppose that the lines are general enough so that all the singularities are nodes. (For instance, while any pair of lines must intersect, three lines should never all meet in a single point.)