DUE DATE: FEB. 7, 2008

1. Find the multiplicative inverse for a mod m for the following a and m:

(a) $a = 13$, $m = 210$

(b) $a = 7$, $m = 15$

(c) $a = 11$, $m = 70$

2. In class we studied how to solve equations of the form $ax \equiv b \pmod{m}$, but we didn't finish one detail: If $d = \gcd(a, m)$ and $d \mid b$ then we saw how to find one solution by solving $a_1 x \equiv b_1 \pmod{m_1}$ where $a_1 = \frac{a}{d}$, $b_1 = \frac{b}{d}$, and $m_1 = \frac{m}{d}$. In class it was claimed that there are $d - 1$ other solutions (for a total of d) which you find in the following way: Let x_0 be the first solution, then all the other solutions are of the form $x_0 + km_1$, where $k = 1, \ldots, d - 1$.

Suppose that $d = \gcd(a, m)$, that $d \mid b$ and that x_0 is a solution to $ax \equiv b \pmod{m}$.

(a) Show that $x_k = x_0 + km_1$ is also a solution to the equation $ax \equiv b \pmod{m}$ for all k, where $m_1 = \frac{m}{d}$.

(b) Show that all solutions are of this form by showing that if x is any solution to $ax \equiv b \pmod{m}$ then $x - x_0 \equiv 0 \pmod{m_1}$ (this shows that the difference is a multiple of m_1, hence is of the form we looked at in (a). HINT: Consider $ax_0 \equiv ax \pmod{m}$ and write this as an equality between integers. Divide both sides of the equality by d and consider the equation mod m_1.)

3. Find all solutions to the following equations:

(a) $13x \equiv 21 \pmod{210}$

(b) $98x \equiv 42 \pmod{210}$

(c) $33x \equiv 5 \pmod{210}$

4. Here is a classic puzzle: Let $A = 4444^{4444}$, let B be the sum of the digits of A, let C be the sum of the digits of B, and finally D the sum of the digits of C.

The problem is: Find D. (You might want to try and puzzle this out on your own for a bit. Afterwards, turn the page ...)

(a) Explain why $A \equiv D \pmod 9$.

(b) Compute $A \pmod 9$ (Hints: (*i*) What is 4444 mod 9? (*ii*) Euler's theorem).

(c) Find an upper bound for the number of possible digits of A. Since each digit is at most nine, use this to find an upper bound on the size of B. [This does not use any ideas from algebra – so use any kind of argument you want to find an upper bound for the number of digits of A]

(d) Repeat this kind of argument to find an upper bound for the sizes of C and D.

(e) Now solve the puzzle: What is D?

5. Here is an elementary card trick. Take fifteen cards out of a deck and ask someone else to secretly pick one of them. Shuffle the fifteen cards as much as you want, then lay them out in this order:

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15

Ask the person to tell you which column their card is in. Pick up the cards again (keeping them all in order) and lay them out like this:

1	4	7	10	13
2	5	8	11	14
3	6	9	12	15

Now have them tell you which row the card is in. After that, you tell them exactly which card they picked. (If you want to make it slightly more dramatic, you can add some elements of psychological misdirection: pick up the cards again in order and have them deal out the cards one by one, telling them that using your psychic powers you will be able to feel the vibrations when they reach the card they've chosen).

The question is: How did you do it? i.e., once you knew the row and column, how did you determine which card was picked? Please give as mathematical an explanation as possible (that is, interpret the trick as a mathematical question, and give precise instructions on how to resolve the question).