DUE DATE: MAR. 6, 2008

1. In order to be able to calculate mod $m(x)$ we need to be able to find remainders. Find the remainder when dividing by $m(x) = x^3 - 6x^2 + 12x - 11$ for the following polynomials in $\mathbb{R}[x]$:

(a) $a(x) = x^3 - 5x^2 + 15x - 9$

(b) $b(x) = 2x^4 - 13x^3 + 31x^2 - 31x + 13$

(c) $c(x) = x^5 - 3x^4 - 10x^2 + 42x - 64$

2. Use the Euclidean algorithm to find the gcd $d(x) = \gcd(a(x), b(x))$ for the following polynomials $a(x)$ and $b(x)$, and then use the extended Euclidean algorithm to find $u(x)$ and $v(x)$ so that $d(x) = a(x)u(x) + b(x)v(x)$.

(a) $a(x) = 3x^2 - 13x + 14$, $b(x) = x^3 - 7x^2 + 14x - 8$ in the ring $\mathbb{Q}[x]$.

(b) $a(x) = \bar{3}x^2 - \overline{13}x + \overline{14}$ $b(x) = x^3 - \bar{7}x^2 + \overline{14}x - \bar{8}$, in the ring $(\mathbb{Z}/5\mathbb{Z})[x]$.

3. The purpose of this question is to develop a divisibility test for the polynomial x^2+1, just like our divisibility for different numbers over the integers.

(a) Show that $x^2 \equiv -1 \pmod{x^2+1}$ in $\mathbb{Q}[x]$.

(b) Show that $x^3 \equiv -x \pmod{x^2+1}$ in $\mathbb{Q}[x]$.

(c) Show that $c_1x + c_0 \equiv 0 \pmod{x^2+1}$ if and only if $(c_1, c_0) = (0, 0)$, where $c_1, c_0 \in \mathbb{Q}$.

(d) Show that a polynomial $a(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_dx^d$ in $\mathbb{Q}[x]$ is divisible by $x^2 + 1$ if and only if

$$\sum_{k\geq 0}(-1)^k a_{2k} = a_0 - a_2 + a_4 - a_6 + \cdots = 0 \quad \text{and}$$

$$\sum_{k\geq 0}(-1)^k a_{2k+1} = a_1 - a_3 + a_5 - a_7 + \cdots = 0.$$

4. Let α be the real number $\alpha = 3^{\frac{1}{3}} + 2$. In parts (a)–(d) calculate the given real numbers to at least six decimal places.

(a) $\alpha^3 - 5\alpha^2 + 15\alpha - 9$

(b) $2\alpha^4 - 13\alpha^3 + 31\alpha^2 - 31\alpha + 13$

(c) $\alpha^5 - 3\alpha^4 - 10\alpha^2 + 42\alpha - 64$

(d) $\alpha^3 - 6\alpha^2 + 12\alpha - 11$.

(e) Now use your answers from question 1 and part (d) to explain what happened in parts (a)–(c).