DUE DATE: MAR. 20, 2008

1. In each of the following examples decide if the set with the given operations is a ring. If the example is not a ring explain what fails. If the example is a ring it is enough to just say so – you don't have to demonstrate that the example satisfies all the axioms.

(a) R is $\mathbb{Z}/2\mathbb{Z}$ except with the operations of addition and multiplication switched. (The new addition is the old multiplication and the new multiplication is the old addition)

(b) R is $\mathbb{R}^2$ with addition $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and multiplication $(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2, x_1y_2 + x_2y_1 + 3y_1y_2)$.

(c) R is the set of infinite sequences $\left\{(a_0, a_1, a_2, \ldots,) \mid a_i \in \mathbb{R},\ \sum_{n=0}^{\infty} |a_n| \text{ converges}\right\}$, where addition is defined as

$$(a_0, a_1, a_2, \ldots,) + (b_0, b_1, b_2, \ldots,) = (a_0 + b_0, a_1 + b_1, a_2 + b_2, \ldots,)$$

and multiplication is defined as

$$(a_0, a_1, a_2, \ldots,) \cdot (b_0, b_1, b_2, \ldots,) = (a_0b_0, a_1b_1, a_2b_2, \ldots,).$$

(d) $R = \{0, 1\}$ with operations

$$x + y = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases} \quad \text{and} \quad x \cdot y = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}.$$

(e) R is $\mathbb{R}^2$ with addition $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and multiplication $(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$.

2. In class it was claimed that the map $\phi\colon \mathbb{Z}/m\mathbb{Z} \longrightarrow \mathbb{Z}/m_1\mathbb{Z}$ given by $a \pmod{m} \mapsto a \pmod{m_1}$ is a ring homomorphism if $m_1 \mid m$. Let's try and figure out what this map is, and see if the condition that $m_1 \mid m$ is necessary.

(a) If $m_1 = 3$ and $m = 5$ give an example of two numbers a and b so that $a \equiv b \pmod{m}$ but $a \not\equiv b \pmod{m_1}$. (NOTE: this is the case $m_1 \nmid m$.)

(b) When we give the instructions "$a \bmod m \mapsto a \bmod m_1$" to describe a map $\mathbb{Z}/m\mathbb{Z} \longrightarrow \mathbb{Z}/m_1\mathbb{Z}$ this means that, for any $\bar{a} \in \mathbb{Z}/m\mathbb{Z}$, pick a representative in the class $\bar{a}$ mod m and look at the class containing this representative mod m_1.

To say that a map is *well defined* means that the result of these instructions does not depend on which representative we pick in the class $\bar{a}$.

When $m_1 = 3$ and $m = 5$, give an example to show that the map above is not well defined.

(c) If $m_1 \mid m$ explain why $a \equiv b \pmod{m}$ implies that $a \equiv b \pmod{m_1}$.

(d) Explain why this means that the map $\phi\colon \mathbb{Z}/m\mathbb{Z} \longrightarrow \mathbb{Z}/m_1\mathbb{Z}$ above is well defined.

3. Suppose that $\phi\colon R \longrightarrow S$ and $\psi\colon S \longrightarrow T$ are ring homomorphisms (where R, S, and T are rings). Show that the composite map $\psi \circ \phi\colon R \longrightarrow T$ is also a ring homomorphism.

4. Describe the kernels of the following ring homomorphisms

(a) $\phi\colon \mathbb{R}[x] \longrightarrow \mathbb{R}$ given by $f(x) \mapsto f(0)$.

(b) $\phi\colon \mathbb{R}[x] \longrightarrow \mathbb{R}$ given by $f(x) \mapsto f(3)$.

(c) $\phi\colon \mathbb{Z}[x] \longrightarrow \mathbb{Z}/5\mathbb{Z}$ given by $f(x) \mapsto f(0) \pmod{5}$.

(d) $\phi\colon \mathbb{Z}[x] \longrightarrow \mathbb{Z}/5\mathbb{Z}$ given by $f(x) \mapsto f(3) \pmod{5}$.

Here "describe" means give a criterion in terms of the coefficients of $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_dx^d$ to decide if f is in the kernel of ϕ or not.

5. In class we learned that we can also describe ideals by generators (e.g. $I = \langle 3, x\rangle \subseteq \mathbb{Z}[x]$). Describe the following ideals by generators (i.e., given an ideal I below, find a_1, …, a_k in the ring R so that $I = \langle a_1, \ldots, a_k\rangle$).

(a) The ideal from 4(a) above.

(b) The ideal from 4(b) above.

(c) The ideal from 4(c) above.

(d) The ideal from 4(d) above.

You should also give a short argument explaining why the elements you claim generate the ideal really generate the ideal. (Hopefully your criteria from question 4 will help…).