Math 210 Homework Assignment 11

DUE DATE: APR. 3, 2008

1. We've used this several times in class, but never written the (short) argument down, so:

(a) In $\mathbb{Z}$ show that $\langle m \rangle \subseteq \langle n \rangle$ iff $n \mid m$.

(b) In $F[x]$ show that $\langle m(x) \rangle \subseteq \langle n(x) \rangle$ iff $n(x) \mid m(x)$.

NOTE: The utility of this result is that it lets us easily figure out which ideals contain a given ideal, at least as long as our ring is a principal ideal domain.

2. For each of the following rings R and ideals I, find all the proper ideals J which contain I. We also want to keep track of inclusion relations among these ideals, so draw a diagram of the ideals you found showing which ones are contained in which others. Here is an example with $R = \mathbb{Z}$ and $I = \langle 12 \rangle$:

$$\begin{tikzcd}
\langle 4 \rangle \arrow[r, dash, "\subset"] & \langle 2 \rangle & \\
\langle 12 \rangle \arrow[u, dash, "\cup"'] \arrow[r, dash, "\subset"] & \langle 6 \rangle \arrow[u, dash, "\cup"'] \arrow[r, dash, "\subset"] & \langle 3 \rangle
\end{tikzcd}$$

i.e., the proper ideals J which contain $\langle 12 \rangle$ are $\{\langle 12 \rangle, \langle 6 \rangle, \langle 4 \rangle, \langle 3 \rangle, \langle 2 \rangle\}$, and the inclusion relations among them are explained by the diagram above.

NOTE 1: It's true that $\langle 12 \rangle \subset \langle 2 \rangle$ but that is implied by the containments $\langle 12 \rangle \subset \langle 6 \rangle$ and $\langle 6 \rangle \subset \langle 2 \rangle$, or the containments $\langle 12 \rangle \subset \langle 4 \rangle$ and $\langle 4 \rangle \subset \langle 2 \rangle$. Not drawing the containment $\langle 12 \rangle \subset \langle 2 \rangle$ directly results in a simpler diagram, and the convention above is to always use such simplifications.

(a) $R = \mathbb{Z}$, $I = \langle 200 \rangle$.

(b) $R = \mathbb{R}[x]$, $I = \langle (x-1)^2(x-5)^2 \rangle$

(c) $R = \mathbb{R}[x]$, $I = \langle x^5 \rangle$.

(d) $R = \mathbb{Z}[x]$, $I = \langle 6, x^2 - 1 \rangle$

NOTE 2: This last question is a bit trickier since the ring $\mathbb{Z}[x]$ is not a principal ideal domain, but in this case things probably work like you think they do.

3. In each of the following quotient rings, list all the ideals in that ring and draw a diagram showing the containment relations. For instance, in the ring $\mathbb{Z}/12\mathbb{Z}$ the ideals and inclusion relations are

$$
\begin{tikzcd}
\langle \bar{4} \rangle \arrow[r, dash, "\subset"] \arrow[d, dash, "\cup"] & \langle \bar{2} \rangle \arrow[d, dash, "\cup"] & \\
\langle \bar{0} \rangle \arrow[r, dash, "\subset"] & \langle \bar{6} \rangle \arrow[r, dash, "\subset"] & \langle \bar{3} \rangle
\end{tikzcd}
$$

In your diagrams you should also indicate which of the ideals are maximal ideals.

(a) $\dfrac{\mathbb{Z}}{\langle 200 \rangle}$.

(b) $\dfrac{\mathbb{R}[x]}{\langle (x-1)^2(x-5)^2 \rangle}$

(c) $\dfrac{\mathbb{R}[x]}{\langle x^5 \rangle}$.

(d) $\dfrac{\mathbb{Z}[x]}{\langle 6, x^2-1 \rangle}$

4. Some questions about the field $\dfrac{\mathbb{Q}[x]}{\langle x^2-2 \rangle}$:

(a) Show or explain why the polynomial x^2-2 is irreducible in $\mathbb{Q}[x]$.

(b) Explain why this means that the quotient ring $\dfrac{\mathbb{Q}[x]}{\langle x^2-2 \rangle}$ is a field (we know many explanations at this point – just pick your favorite).

(c) Since the quotient ring is a field, every nonzero element has a multiplicative inverse. Find the multiplicative inverse $p(x)$ of the class of $2+3x$ in the quotient ring. (i.e., find a polynomial $p(x) \in \mathbb{Q}[x]$ such that $(2+3x) \cdot p(x) \equiv 1 \pmod{x^2-2}$).

(d) Compute the numbers $\frac{1}{2+3\sqrt{2}}$ and $p(\sqrt{2})$ to at least six decimal places, and explain the result.

(e) Is the ring $\left\{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\right\}$ a field? Why? How do you find the multiplicative inverse for a nonzero element? (The last question has many possible answers).