Chapter 10: Compact Metric Spaces

10.1 Definition. A collection of open sets \( \{U_i : i \in I\} \) in \( X \) is an open cover of \( Y \subset X \) if \( Y \subset \bigcup_{i \in I} U_i \). A subcover of \( \{U_i : i \in I\} \) is a subcollection \( \{U_j : j \in J\} \) for some \( J \subset I \) that still covers \( Y \). It is a finite subcover if \( J \) is finite.

10.2 Definition.

1. A metric space \( X \) is compact if every open cover of \( X \) has a finite subcover.

2. A metric space \( X \) is sequentially compact if every sequence of points in \( X \) has a convergent subsequence converging to a point in \( X \).

10.3 Examples.

1. \((0, 1]\) is not sequentially compact (using the Heine-Borel theorem) and not compact. To show that \((0, 1]\) is not compact, it is sufficient find an open cover of \((0, 1]\) that has no finite subcover. But a moment’s consideration of the cover consisting exactly of the sets \( U_n := \left(\frac{1}{n}, 2\right) \) shows that this is just such a cover. That is, \( \bigcup_{n \in \mathbb{N}} U_n = (0, 2) \supset (0, 1] \).

2. \([0, 1]\) is sequentially compact (applying Heine-Borel). In fact, \([0, 1]\) is also compact (as we will see shortly).

3. \(\mathbb{R}\) is neither compact nor sequentially compact. That it is not sequentially compact follows from the fact that \(\mathbb{R}\) is unbounded and Heine-Borel. To see that it is not compact, simply notice that the open cover consisting exactly of the sets \( U_n = (-n, n) \) can have no finite subcover. Using reasoning similar to that of example 1, if \( F \) is a finite subset of \( \{U_n : n \in \mathbb{N}\} \) then \( F \) contains an element \( U_k \) such that \( k \geq i \) for each \( U_i \in F \). But this means that \( U_k \supset U_i \) for each \( U_i \in F \) and hence \( \bigcup F = U_k = \left(\frac{1}{2}, 2\right) \subset (0, 1]\).

4. \(X = C[0, 1]\) is not compact. Denote by \( B_r(f) \) the open ball of radius \( r \) under the sup-norm centered at the function \( f \in C[0, 1] \), and consider
the set $U = \{ B_{\frac{1}{4}}(f) : f \in C[0,1] \}$. Clearly, $U$ is an open cover of $C[0,1]$. Now define the sequence $(f_n)_{n \in \mathbb{N}}$ as

$$f_n(x) = \begin{cases} 
0 & \text{if } 0 \leq x \leq \frac{1}{n+1} \\
2n(n+1)(x - \frac{1}{n+1}) & \text{if } \frac{1}{n+1} < x \leq \frac{2n+1}{2n(n+1)} \\
-2n(n+1)(x - \frac{1}{n}) & \text{if } \frac{2n+1}{2n(n+1)} < x \leq \frac{1}{n} \\
0 & \text{if } \frac{1}{n} < x \leq 1 
\end{cases}$$

The graphs of $f_1, f_2, f_3,$ and $f_4$ are plotted below.

Notice that $\| f_m - f_n \|_{\infty} = 1$ whenever $m \neq n$. Hence, each element of $U$ can contain at most one function from $(f_n)_{n \in \mathbb{N}}$, and therefore every finite subset of $U$ fails to cover $C[0,1]$.

5. (Cantor’s Intersection Theorem.) If $C_1 \supset C_2 \supset C_3 \ldots$ is a decreasing sequence of nonempty sequentially compact subsets of $\mathbb{R}^n$, then $\cap_{k \geq 1} C_k$ is non-empty.

To see this, choose the sequence $(a_n)_{n \in \mathbb{N}}$ so that $a_n \in C_n$ for every $n$. Clearly, $(a_n)$ is a sequence in $C_1$. The compactness of $C_1$ tells us that $(a_n)$ has a convergent subsequence $(a_{n_k})_{k \in \mathbb{N}}$. Say that $a_{n_k} \to a$ as $k \to \infty$. Then $a$ is the limit of a sequence in $C_i$ for each $i$, which means that $a \in C_i$ for each $i$, and the result follows.

10.4 Definition. A metric space $X$ is totally bounded if, for every $\epsilon > 0$, there exist $x_1, x_2, \ldots, x_k \in X$, with $k$ finite, so that $\{ B_\epsilon(x_i) : 1 \leq i \leq k \}$ is an open cover of $X$. 
10.5 Examples.

1. Example 4 of 10.3 shows that the closed unit ball in \( C[0, 1] \) is not totally bounded.

2. \((0, 1]\) is totally bounded since for any \( \epsilon > 0 \), \( \{(i\epsilon, (i + 1)\epsilon) : 0 \leq i \leq \frac{1}{\epsilon}, i \in \mathbb{Z}\} \) is an open cover. However, \((0, 1]\) is not compact. (We will see shortly that the ingredient missing from \((0, 1]\) and essential for compactness is in fact completeness.)

10.6 Definition. A collection of closed sets \( \{C_i : i \in I\} \) has the finite intersection property if every finite subcollection has nonempty intersection.

10.7 Theorem. (The Borel-Lebesgue Theorem.) For a metric space \((X, \rho)\), the following are equivalent:

1. \(X\) is compact.
2. Every collection of closed subsets of \(X\) with the finite intersection property has non-empty intersection.
3. \(X\) is sequentially compact.
4. \(X\) is complete and totally bounded.

Proof. We’ll first show (1) implies (2). Consider a collection of closed subsets \( \{C_i : i \in I\} \) of \(X\) having the finite intersection property, and assume that \( \bigcap_{i \in I} C_i = \emptyset \). Put \( U_i := C_i^c \) for each \( i \), and notice that each \( U_i \) is open. We have \( \bigcup_{i \in I} U_i = \bigcup_{i \in I} C_i^c = (\bigcap_{i \in I} C_i)^c = X \), so that \( \{U_i : i \in I\} \) is an open cover of \(X\). Since \(X\) is compact, there exists a finite subcover \( \{U_{n_1}, U_{n_2}, \ldots, U_{n_k}\} \) of \(X\). Hence,

\[
X = U_{n_1} \cup U_{n_2} \cup \ldots \cup U_{n_k} \\
= (U_{n_1}^c \cap U_{n_2}^c \cap \ldots \cap U_{n_k}^c)^c \\
= (C_{n_1} \cap C_{n_2} \cap \ldots \cap C_{n_k})^c,
\]

which means that \( C_{n_1} \cap C_{n_2} \cap \ldots \cap C_{n_k} = \emptyset \), in contradiction with the finite intersection property.

The argument from (2) to (1) runs as follows: Suppose that \( \{U_i : i \in I\} \)
I\} is an open cover of X and put $C_i := U_i^c$ for each $i$. Suppose further that no finite subset of \{U_i : i \in I\} covers X. Then if a subcollection \{C_{n_1}, C_{n_2}, \ldots, C_{n_k}\} of \{C_i : i \in I\} satisfies $C_{n_1} \cap C_{n_2} \cap \ldots \cap C_{n_k} = \emptyset$, we would have

$$U_{n_1} \cup U_{n_2} \cup \ldots \cup U_{n_k}$$

$$= (U_{n_1}^c \cap U_{n_2}^c \cap \ldots \cap U_{n_k}^c)^c$$

$$= (C_{n_1} \cap C_{n_2} \cap \ldots \cap C_{n_k})^c$$

$$= X,$$

a contradiction with the assumption that no finite subset of \{U_i : i \in I\} covers X. Thus, \{C_i : i \in I\} has the finite intersection property. By (2), $\cap_{i \in I} C_i \neq \emptyset$, so $\cup_{i \in I} U_i \neq X$, meaning that \{U_i : i \in I\} is not an open cover for X.

We’ll now show that (3) follows from (1). Assume we have a sequence $(x_n)_{n \in \mathbb{N}}$ in X with no convergent subsequence. Since no term in the sequence can occur infinitely many times (otherwise we would have a convergent subsequence), we can assume without loss of generality that $x_i \neq x_j$ whenever $i \neq j$. Notice that each term of the sequence $(x_n)$ is an isolated point of \{x_n : n \in \mathbb{N}\}, since otherwise, $(x_n)$ would have a convergent subsequence. Hence, for each $i$ there exists an open ball, call it $U_i$, centred at $x_i$ with the property that $x_j \notin U_i$ for all $i \neq j$. Now put $U_0 := X \setminus \{x_n : n \in \mathbb{N}\}$. $U_0$ is open since its complement consists only of isolated points, and so is closed. Then \{U_0\} $\cup$ \{U_n : n \in \mathbb{N}\} is an open cover for X. But this open cover has no finite subcover, since any finite subcollection of these sets would fail to include infinitely many terms from the sequence $(x_n)$ in its union.

To see that (3) implies (4), assume that $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in a sequentially compact space X. Say that $(x_{n_k})_{k \in \mathbb{N}}$ is a convergent subsequence of $(x_n)$ and that $x_{n_k} \to x$. Let $\epsilon > 0$ be given, and choose $N$ so that $\rho(x_i, x_j) < \epsilon/2$ whenever $i, j \geq N$. Next, choose $n_k > N$ so that $\rho(x_{n_k}, x) < \epsilon/2$. Then we have

$$\rho(x, x_N) \leq \rho(x, x_{n_k}) + \rho(x_{n_k}, x_N) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus, $x_n \to x$ as $n \to \infty$, showing that X is complete. Next, we’ll show that X is totally bounded.

Assume that X is not totally bounded, and take $\epsilon > 0$ such that X cannot be covered by a collection consisting of only finitely many $\epsilon$-balls.
Choose \( x_1 \in X \), \( x_2 \in X \setminus B_\epsilon(x_1) \), then \( x_3 \in X \setminus B_\epsilon(x_1) \setminus B_\epsilon(x_2) \), and so on. We thus have a sequence \( (x_n) \) which cannot contain a convergent subsequence since \( \rho(x_i, x_j) \geq \epsilon \) for all \( i \neq j \).

We can also obtain (3) from (4). Consider a sequence \( (x_n)_{n \in \mathbb{N}} \) in \( X \). Since \( X \) is totally bounded, we have, for every \( n \in \mathbb{N} \), a finite set of points \( \{y_1^{(n)}, y_2^{(n)}, \ldots, y_{r(n)}^{(n)}\} \) such that \( X \subset B_\frac{1}{n}(y_1^{(n)}) \cup \ldots \cup B_\frac{1}{n}(y_{r(n)}^{(n)}) \). Let \( (S_n)_{n \in \mathbb{N}} \) be the sequence of finite subsets of \( X \) obtained by putting \( S_n := \{y_1^{(n)}, \ldots, y_{r(n)}^{(n)}\} \). We can find a convergent subsequence \( (z_n)_{n \in \mathbb{N}} \) of \( (x_n) \) using the following procedure: Since \( S_1 \) is finite, there is a \( y_{n(1)}^{(1)} \in S_1 \) such that \( B_1(y_{n(1)}^{(1)}) \) contains infinitely many points from \( (x_n) \). Select \( z_1 \) from \( B_1(y_{n(1)}^{(1)}) \). Now, since \( S_2 \) is finite, there is a \( y_{n(2)}^{(2)} \in S_2 \) such that \( B_1(y_{n(1)}^{(1)}) \cap B_\frac{1}{2}(y_{n(2)}^{(2)}) \) contains infinitely many points from \( (x_n) \). Choose \( z_2 \) from \( B_1(y_{n(1)}^{(1)}) \cap B_\frac{1}{2}(y_{n(2)}^{(2)}) \). Now continue this procedure for each \( k > 1 \), selecting \( y_{n(k)}^{(k)} \) from \( S_k \) such that \( \cap_{i=1}^{k} B_\frac{1}{i}(y_{n(i)}^{(i)}) \) contains infinitely many points from \( (x_n) \), and then selecting \( z_k \) from \( \cap_{i=1}^{k} B_\frac{1}{i}(y_{n(i)}^{(i)}) \). \( (z_n) \) is clearly Cauchy, and by the completeness of \( X \), \( (z_n) \) converges to a point in \( X \).

Finally, we show that (3) implies (1). We’ll need the following preliminary result: If \( (X, \rho) \) is a sequentially compact metric space, and \( \{U_i : i \in I\} \) an open cover for \( X \). Then there is an \( r > 0 \) such that for each \( x \in X \), \( B_r(x) \subseteq U_i \) for some \( i \in I \). The proof is by contradiction. Assume that for some \( r > 0 \), there is an \( x \in X \), possibly depending on \( r \), such that for each \( i \in I \), \( B_r(x) \not\subseteq U_i \). Now choose the sequence \( (x_n)_{n \in \mathbb{N}} \) in \( X \) so that \( B_\frac{1}{n}(x_n) \not\subseteq U_i \) for all \( i \in I \).

Since \( X \) is sequentially compact, \( (x_n) \) has a convergent subsequence, \( (x_{n_k})_{k \in \mathbb{N}} \). Say that \( x_{n_k} \rightharpoonup x \) as \( k \to \infty \), where \( x \in X \). There must be some \( i_0 \) such that \( x \in U_{i_0} \), and, since \( U_{i_0} \) is open, an \( r_0 > 0 \) such that \( B_{r_0}(x) \subseteq U_{i_0} \). So choose \( N \) such that \( \rho(x, x_N) < \frac{1}{2}r_0 \) and \( \frac{1}{N} < \frac{1}{2}r_0 \). Now if \( y \in B_\frac{1}{N}(x_N) \), then

\[
\rho(x, y) \leq \rho(x, x_N) + \rho(x_N, y) \\
< \frac{1}{2}r_0 + \frac{1}{2}r_0 \\
< r_0,
\]

and hence \( y \in B_{r_0}(x) \subseteq U_{i_0} \). It follows that \( B_\frac{1}{N}(x_N) \subseteq B_{r_0}(x) \subseteq U_{i_0} \), a contradiction, and the preliminary result is proven.
Returning to the main argument, let \( \{U_i : i \in I\} \) be an open cover of \( X \). By the preliminary result, there exists an \( r > 0 \) such that for each \( x \in X \), \( B_r(x) \subset U_i \) for some \( i \in I \). We know that (3) implies (4), thus \( X \) is totally bounded. That is, \( X \subset B_r(y_1) \cup B_r(y_2) \cup \ldots \cup B_r(y_k) \) for some points \( y_1, y_2, \ldots, y_k \in X \), with \( k \in \mathbb{N} \). However, for each \( i \in I \), we have that \( B_r(y_i) \subset U_{j(i)} \) for some \( j(i) \in I \). Thus \( \{U_{j(1)}, U_{j(2)}, \ldots, U_{j(k)}\} \) is a finite subcover for \( X \), and we’re done. \( \square \)

10.8 Theorem. Let \((X, \zeta)\) and \((Y, \delta)\) be metric spaces and \( f : X \to Y \) be a continuous function. Then for each compact subset \( C \subset X \), \( f(C) \subset Y \) is compact.

Proof. Let \( \{U_i : i \in I\} \) be an open cover of \( f(C) \), and for each \( i \in I \), define \( V_i \) to be the pre-image of \( U_i \) under \( f \). Notice that since \( f \) is continuous, each \( V_i \) is open. Thus, \( \{V_i : i \in I\} \) is an open cover of \( C \). But since \( C \) is compact, there exists a finite subcover \( \{V_{i(1)}, V_{i(2)}, \ldots, V_{i(k)}\} \) for \( C \), and hence \( \{U_{i(1)}, U_{i(2)}, \ldots, U_{i(k)}\} \) is a finite subcover of \( f(C) \). So, \( f(C) \) is compact. \( \square \)