# Quantum groups and liberation of orthogonal matrix groups

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joint work with Teodor Banica

## Liberation

group  $\longrightarrow$  quantum group

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orthogonal group  $\longrightarrow$  quantum orthogonal group

permutation group  $\longrightarrow$  quantum permutation group

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???	$\longrightarrow$	quantum ???
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#### **Orthogonal Hopf Algebra**

is a  $C^*$ -algebra A, given with a system of  $n^2$  self-adjoint generators  $u_{ij} \in A$  (i, j = 1, ..., n), subject to the following conditions:

- The inverse of  $u = (u_{ij})$  is the transpose matrix  $u^t = (u_{ji})$ .
- $\Delta(u_{ij}) = \Sigma_k u_{ik} \otimes u_{kj}$  defines a morphism  $\Delta : A \to A \otimes A$ .
- $\varepsilon(u_{ij}) = \delta_{ij}$  defines a morphism  $\varepsilon : A \to \mathbb{C}$ .
- $S(u_{ij}) = u_{ji}$  defines a morphism  $S : A \to A^{op}$ .

These are compact quantum groups in the sense of Woronowicz.

## Quantum Orthogonal Group (Wang 1995)

The quantum orthogonal group  $A_o(n)$  is the universal unital  $C^*$ algebra generated by  $u_{ij}$  (i, j = 1, ..., n) subject to the relation

• 
$$u = (u_{ij})_{i,j=1}^n$$
 is an orthogonal matrix

This means: for all i, j we have

$$\sum_{k=1}^{n} u_{ik} u_{jk} = \delta_{ij} \qquad \text{and} \qquad$$

$$\sum_{k=1}^{n} u_{ki} u_{kj} = \delta_{ij}$$

## **Quantum Permutation Group (Wang 1998)**

The quantum permutation group  $A_s(n)$  is the universal unital  $C^*$ algebra generated by  $u_{ij}$  (i, j = 1, ..., n) subject to the relations

• 
$$u_{ij}^2 = u_{ij} = u_{ij}^*$$
 for all  $i, j = 1, ..., n$ 

• each row and column of  $u = (u_{ij})_{i,j=1}^n$  is a partition of unity:

$$\sum_{j=1}^{n} u_{ij} = 1 \qquad \sum_{i=1}^{n} u_{ij} = 1$$

(this will feature prominently in the talk of Claus Koestler!)

## How can we describe and understand intermediate quantum groups, sitting between these two cases:

$$A_o(n) \to \mathbf{A} \to A_s(n)$$

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## Deal with quantum groups by looking on their representations!!!

#### **Spaces of Intertwiners**

Associated to an orthogonal Hopf algebra  $(A, (u_{ij})_{i,j=1}^n)$  are the spaces of intertwiners:

$$C_a(k,l) = \{T : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l} \mid Tu^{\otimes k} = u^{\otimes l}T\}$$

where  $u^{\otimes k}$  is the  $n^k \times n^k$  matrix  $(u_{i_1j_1} \dots u_{i_kj_k})_{i_1 \dots i_k, j_1 \dots j_k}$ .

$$u \in M_n(A)$$
  $u : \mathbb{C}^n \to \mathbb{C}^n \otimes A$ 

$$u^{\otimes k}: (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes k} \otimes A^{\otimes k} \cong (\mathbb{C}^n)^{\otimes k} \otimes A$$

## **Tensor Category with Duals**

The collection of vector spaces  $C_a(k, l)$  has the following properties:

- $T, T' \in C_a$  implies  $T \otimes T' \in C_a$ .
- If  $T, T' \in C_a$  are composable, then  $TT' \in C_a$ .
- $T \in C_a$  implies  $T^* \in C_a$ .
- id(x) = x is in  $C_a(1, 1)$ .
- $\xi = \sum e_i \otimes e_i$  is in  $C_a(0,2)$ .

#### **Quantum Groups** $\leftrightarrow$ **Intertwiners**

The compact quantum group A can actually be rediscovered from its space of interwiners:

There is a one-to-one correspondence between:

- orthogonal Hopf algebras  $A_o(n) \rightarrow \mathbf{A} \rightarrow A_s(n)$
- tensor categories with duals  $C_{ao} \subset \mathbf{C}_{\mathbf{a}} \subset C_{as}$ .

We denote by P(k, l) the set of partitions of the set with repetitions  $\{1, \ldots, k, 1, \ldots, l\}$ . Such a partition will be pictured as

$$p = \begin{cases} 1 \dots k \\ \mathcal{P} \\ 1 \dots l \end{cases}$$

where  $\ensuremath{\mathcal{P}}$  is a diagram joining the elements in the same block of the partition.

Example in P(5, 1):

Associated to any partition  $p \in P(k, l)$  is the linear map

$$T_p: (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l}$$

given by

$$T_p(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \dots j_l} \delta_p(i, j) e_{j_1} \otimes \ldots \otimes e_{j_l}$$

where  $e_1, \ldots, e_n$  is the standard basis of  $\mathbb{C}^n$ .

$$T_{-}\left\{ \left| \right. \right| \right\} (e_a \otimes e_b) = e_a \otimes e_b$$

$$T_{-}\left\{\left| -\right|\right\} (e_a \otimes e_b) = \delta_{ab} e_a \otimes e_a$$

$$T_{-} \left\{ \begin{matrix} \sqcup \\ \mid \end{matrix} \right\} (e_a \otimes e_b) = \delta_{ab} \sum_{cd} e_c \otimes e_d$$

## Intertwiners of Quantum Permutation and of Quantum Orthogonal Group

Let  $NC(k, l) \subset P(k, l)$  be the subset of noncrossing partitions.

The tensor category of  $A_s(n)$  is given by:

$$C_{as}(k,l) = \operatorname{span}(T_p | p \in NC(k,l))$$

The tensor category of  $A_o(n)$  is given by:

 $C_{ao}(k,l) = \operatorname{span}(T_p | p \in NC_2(k,l))$ 

## **Free Quantum Groups**

A quantum group  $A_o(n) \rightarrow \mathbf{A} \rightarrow A_s(n)$  is called **free** when its associated tensor category is of the form

$$C_{as} = \operatorname{span}(T_p \mid p \in NC)$$

$$\cup$$

$$C_a \qquad \qquad \cup$$

$$C_{ao} = \operatorname{span}(T_p \mid p \in NC_2)$$

## **Free Quantum Groups**

A quantum group  $A_o(n) \rightarrow \mathbf{A} \rightarrow A_s(n)$  is called **free** when its associated tensor category is of the form

$$C_{as} = \operatorname{span}(T_p \mid p \in NC)$$

$$\cup$$

$$C_a = \operatorname{span}(T_p \mid p \in NC_a),$$

$$\cup$$

$$C_{ao} = \operatorname{span}(T_p \mid p \in NC_2)$$

for a certain collection of subsets  $NC_a \subset NC$ .

## **Category of Noncrossing Partitions**

A category of noncrossing partitions is a collection of subsets  $NC_x(k,l) \subset NC(k,l)$ , subject to the following conditions:

- $NC_x$  is stable by tensor product.
- $NC_x$  is stable by composition.
- $NC_x$  is stable by involution.
- $NC_x$  contains the "unit" partition |.
- $NC_x$  contains the "duality" partition  $\Box$ .

## Category of Noncrossing Partitions ↔ Free Quantum Groups

Let  $NC_x$  be a category of noncrossing partitions, and  $n \in \mathbb{N}$ .

- $C_x = \operatorname{span}(T_p | p \in NC_x)$  is a tensor category with duals.
- The associated quantum group  $A_o(n) \rightarrow A \rightarrow A_s(n)$  is free.
- Any free quantum group appears in this way.

## There are 6 Categories of Noncrossing Partitions:

$$\begin{cases} \text{singletons and} \\ \text{pairings} \end{cases} \supset \begin{cases} \text{singletons and} \\ \text{pairings (even part)} \end{cases} \supset \begin{cases} \text{all} \\ \text{pairings} \end{cases}$$
$$\cap \qquad \cap \qquad \\ \\ \begin{cases} \text{all} \\ \text{partitions} \end{cases} \supset \begin{cases} \text{all partitions} \\ (\text{even part)} \end{cases} \supset \begin{cases} \text{with blocks of} \\ \text{even size} \end{cases}$$

## ... and thus 6 free Quantum Groups:

$$egin{array}{rcl} A_b(n) &\leftarrow& A_{b'}(n) &\leftarrow& A_o(n) \ &\downarrow && \downarrow && \downarrow \ A_s(n) &\leftarrow& A_{s'}(n) &\leftarrow& A_h(n) \end{array}$$

- Orthogonal, if its entries are self-adjoint, and  $uu^t = u^t u = 1$ .
- *Magic*, if it is orthogonal, and its entries are projections.
- Cubic, if it is orthogonal, and  $u_{ij}u_{ik} = u_{ji}u_{ki} = 0$ , for  $j \neq k$ .
- Bistochastic, if it is orthogonal, and  $\Sigma_j u_{ij} = \Sigma_j u_{ji} = 1$ .
- *Magic'*, if it is cubic, with the same sum on rows and columns.
- *Bistochastic'*, if it is orthogonal, with the same sum on rows and columns.

## **More General Classification**





 $A_o(n) \rightarrow A_{free} \rightarrow A_s(n)$   $\downarrow \qquad \qquad \downarrow$   $C(O_n) \rightarrow C(S_n)$ 

• There are exactly six free quantum groups  $A_{free}!$ 



 $A_o(n) \rightarrow A_{free} \rightarrow A_s(n)$   $\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$   $C(O_n) \rightarrow C(G_{easy}) \rightarrow C(S_n)$ 

- There are exactly six free quantum groups  $A_{free}!$
- There are exactly six classical easy groups  $G_{easy}!$

## **More General Classification**



- There are exactly six free quantum groups  $A_{free}!$
- There are exactly six classical easy groups  $G_{easy}!$
- Can we have more easy quantum groups A<sub>easy</sub>???

A quantum group satisfying  $A_o(n) \rightarrow A \rightarrow C(S_n)$  is called **easy** when its associated tensor category of intertwiners is spanned by partitions. The corresponding **full category of partitions**  $P_x \subset P$  must satisfy:

- $P_x$  is stable by tensor product.
- $P_x$  is stable by composition.
- $P_x$  is stable by involution.
- $P_x$  contains the "unit" partition |.
- $P_x$  contains the "duality" partition  $\Box$ .

## There are at least 3 more easy quantum groups

The following are full categories of partitions:

- $P_o^*$ : the set of pairings having the property that each string has an even number of crossings.
- $P_b^*$ : the set of singletons and pairings having the property that when removing the singletons, each string has an even number of crossings.
- $P_{b'}^*$ : the even part of  $P_b^*$ , consisting of pairings having an even number of crossings, completed with an even number of singletons.

#### $P_o^*$ is generated by



The algebras  $A_o^*(n), A_b^*(n), A_{b'}^*(n)$  are respectively the quotients of the algebras  $A_o(n), A_b(n), A_{b'}(n)$  by the collection of relations

$$abc = cba$$

one for each choice of a, b, c in the set  $\{u_{ij} | i, j = 1, ..., n\}$ .

#### Literature

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