Spectra and energy of signed digraphs

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Spectra and energy of graphs and digraphs

In this section, we give a brief introduction of energy of graphs and digraphs. We start with the adjacency matrix of a graph. Let G be a graph with n vertices $v_1, v_2, \cdots, v_n$ and m edges. The adjacency matrix of G is the $n \times n$ matrix $A(G) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & \text{if there is an edge from } v_i \text{ to } v_j, \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial $det(xI - A(G))$ of the adjacency matrix $A(G)$ of G is called the characteristic polynomial of G and is denoted by $\phi_G(x)$. The eigenvalues of $A(G)$ are called the eigenvalues of G. The set of distinct eigenvalues of G together with their multiplicities is called the spectrum of G. If G has k distinct eigenvalues $x_1, x_2, \cdots, x_k$ with respective multiplicities $m_1, m_2, \cdots, m_k$, then we write the spectrum of G as $spec(G) = \{x_1^{(m_1)}, x_2^{(m_2)}, \cdots, x_k^{(m_k)}\}$.

Recall a graph is said to be an elementary figure if it is either K_2 or a cycle C_q, where $q \geq 3$. A basic figure is a graph whose components are elementary figures. The following result relates the coefficients of the characteristic polynomial of a graph with the structure of the graph and is also known as Sach's Theorem [?].

Theorem A$_1$

Let G be a graph of order n and with characteristic polynomial

$$\phi_G(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1}x + a_n.$$

Then

$$a_j = \sum_{L \in \mathscr{L}_j} (-1)^{p(L)} 2^{|c(L)|},$$

for all $j = 1, 2, \cdots, n$, where $\mathscr{L}_j$ is the set of all basic figures L of G of order j, $p(L)$ denotes number of components of L and $c(L)$ denotes the set of all cycles of L.

A graph is bipartite if and only if it contains no odd cycles. As basic figures of odd order must possess at least one odd cycle, therefore for a bipartite graph $\mathscr{L}_{2j+1} = \emptyset$ for all $j \geq 0$ and hence $a_{2j+1} = 0$ for all $j \geq 0$. Consequently, the characteristic polynomial of a bipartite graph B takes the form

$$\phi_B(x) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} a_{2j} x^{n-2j}.$$

The even coefficients of a bipartite graph alternate in sign [?] i.e., $(-1)^j a_{2j} \geq 0$ for all j. Therefore

$$\phi_B(x) = x^n + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^j b_{2j} x^{n-2j}, \tag{1}$$

where $a_{2j} = (-1)^j b_{2j}$ and b_{2j} are non negative integers.

Energy of graphs

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Definition 1.2 Let G be a graph of order n with eigenvalues $x_1, x_2, \ldots, x_n$. The energy of G is defined as

$$E(G) = \sum_{j=1}^{n} |x_j|.$$

This concept was given by Gutman [**?**] in 1978. The following is the integral representation for the energy of a graph (also known as Coulson's integral formula)

Theorem B$_1$

Let G be a graph with n vertices having characteristic polynomial $\phi_G(x)$. Then

$$E(G) = \sum_{j=1}^{n} |x_j| = \frac{1}{\pi} \int_{-\infty}^{\infty} (n - \frac{\iota x \phi'_G(\iota x)}{\phi_G(\iota x)}) dx,$$

where $x_1, x_2, \cdots, x_n$ are the eigenvalues of graph G, $\iota = \sqrt{-1}$ and $\int_{-\infty}^{\infty} F(x)dx$ denotes the principle value of the respective integral.

The following observations [?] follow from Coulson's integral formula.

Theorem C_1

If G is a graph of order n, then

$$E(G) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{x^2} \log |x^n \phi_G(\frac{\iota}{x})| dx.$$

Combining (1) and Theorem C_1, the energy of a bipartite graph B is given as under.

Theorem D_1

If B is a bipartite graph on n vertices, then

$$E(B) = \frac{2}{\pi} \int_{0}^{\infty} \frac{1}{x^2} \log[1 + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} b_{2j} x^{2j}] dx,$$

where $b_{2j} \geq 0$.

From this energy expression, we see that the energy of a bipartite graph is an increasing function of the coefficients b_{2j}. Given bipartite graphs B_1 and B_2 (not of same order), we say $B_1 \preceq B_2$ if and only if $b_{2j}(B_1) \leq b_{2j}(B_2)$. If for some j, $b_{2j}(B_1) < b_{2j}(B_2)$, then we say $B_1 \prec B_2$. Thus the relation $\preceq$ is a quasi-order relation (i.e., reflexive and transitive) and energy increases with respect to this relation. That is, if $B_1 \prec B_2$ then $E(B_1) < E(B_2)$.

Bounds for the energy of a graph.
Several upper and lower bounds for the energy are known. The following upper and lower bound of energy of a graph in terms of order n, size m and determinant of adjacency matrix is due to McClelland [**?**].

Theorem E_1

If G is a graph with n vertices and m edges, then

$$\sqrt{2m + n(n-1)|det(A(G))|^{\frac{2}{n}}} \leq E(G) \leq \sqrt{2mn}. \tag{2}$$

The graph energy as a function of the number of edges satisfies the following inequalities [**?**].

Theorem F_1

If G is a graph with m edges, then

$$2\sqrt{m} \leq E(G) \leq 2m.$$

with equality on the left if and only if G is a complete bipartite graph plus some isolated vertices and equality on the right if and only if G is a matching of m edges plus some isolated vertices.

The following is a lower bound for the energy of a graph in terms of its number of vertices.

Theorem G_1

If G is a graph with n vertices, then

$$E(G) \geq 2\sqrt{n-1}$$

with equality if and only if $G = K_{1,n-1}$.

Equienergetic graphs

Two graphs G_1 and G_2 are said to be cospectral if they have same spectrum and non-cospectral, otherwise. Isomorphic graphs are cospectral, since adjacency matrices of isomorphic graphs are similar by means of a permutation matrix. There exist non isomorphic cospectral graphs [**?**]. Cospectral graphs are obviously equienergetic, therefore problem of equienergetic graphs reduces to the problem of construction of non-cospectral equienergetic graphs.

Line graph and iterated line graph. The line graph $L(G)$ of a graph G is the graph whose vertex set is the edge set of G and any two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G share a vertex.
Given a graph G, let $L^1(G) = L(G), L^2(G) = L(L(G)), \cdots, L^k(G) = L(L^{k-1}(G))$. Then $L^k(G)$ is called the k-th iterated line graph of G.

Theorem H_1

If G is an $r(\geq 3)$-regular graph of order n, then

$$E(L^2(G)) = 2nr(r-2).$$

From Theorem H_1 and noting that iterated line graphs of non cospectral regular graphs are non cospectral, the following result [?] yields the existence of non cospectral equienergetic graphs.

Theorem I_1

Let G_1 and G_2 be two non cospectral regular connected graphs both on n vertices and both of degree $r \geq 3$. Then $L^2(G_1)$ and $L^2(G_2)$ are connected, non cospectral and equienergetic.

Balakrishnan [?] proved that for a non trivial graph Q, if $G = C_4$ and $H = K_2 \otimes K_2$, then $Q \otimes G$ and $Q \otimes H$ are non cospectral and equienergetic. Bonifacio et al. [?] have given conditions on an arbitrary pair G and H of equienergetic non cospectral graphs to make assertion true for any non trivial connected graph Q.

Theorem J_1

Let G and H be two equienergetic non cospectral graphs such that there is an eigenvalue x of G for which $x > |y|$, for all eigenvalues y of H. If Q is a non trivial connected graph, then $Q \otimes G$ and $Q \otimes H$ are equienergetic non cospectral graphs.

Hyperenergetic graphs

From Theorem E_1, a graph G with n vertices and m edges satisfies the upper bound $E(G) \leq \sqrt{2mn}$. This bound depends only on m and n. As among all n-vertex graphs, the complete graph K_n has maximum number of edges which is $\frac{n(n-1)}{2}$. This motivated Gutman to conjecture that among all n-vertex graphs, the complete graph K_n has maximum energy which is equal to $2(n-1)$. Later Godsil [**?**] in 1980's proved that there exists graphs of order n with energy greater than $2(n-1)$. This motivated the following definition.

Hyperenergetic graph. A graph G of order n is said to be hyperenergetic if $E(G) > 2(n-1)$. Gutman et al. [**?**] proved that no Hückel graph (molecular graph) is hyperenergetic. Panigrahi and Mohapatra [**?**] proved all primitive strongly regular graphs except SRG$(5,2,0,1)$, SRG$(9,4,1,2)$, SRG$(10,3,0,1)$and SRG$(16,5,0,2)$ are hyperenergetic.

Energy of digraphs

Peña and Rada [**?**] extended the concept of energy to digraphs in such a way that Coulson's integral formula remains valid. Before defining energy of a digraph, we give a brief introduction of spectra of digraphs.

Let D be a digraph with n vertices $v_1, v_2, \cdots, v_n$. The adjacency matrix of D is the $n \times n$ matrix $A(D) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & \text{if there is an arc from } v_i \text{ to } v_j, \\ 0, & \text{otherwise.} \end{cases}$$

Unlike graphs the adjacency matrix of a digraph need not be real symmetric, so eigenvalues can be complex numbers. We denote the characteristic polynomial $det(xI - A(D))$ of the adjacency matrix $A(D)$ by $\phi_D(x)$. If $z_1, z_2, \ldots, z_n$ are eigenvalues of digraph D, we label them so that $\Re z_1 \geq \Re z_2 \geq \cdots \geq \Re z_n$, where $\Re z_j$ denotes the real part of complex number z_j. By Perron Frobenius theorem $\Re z_1$ is an eigenvalue of D with largest absolute value and is called spectral radius of D. It is denoted by ρ. A linear subdigraph of a digraph D is a subdigraph with indegree and outdegree of each vertex equal to one. Consequently, a linear subdigraph is either a cycle or disjoint union of cycles.

The following result due to Sach [?] relates the coefficients of the characteristic polynomial of a digraph with its structure.

Theorem K_1

If D is a digraph of order n with characteristic polynomial

$$\phi_D(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n,$$

then

$$a_j = \sum_{L \in \$_j} (-1)^{p(L)},$$

where $\$_j$ is the set of all linear subdigraphs of D of order j and $p(L)$ denotes the number of components of L.

The following is the definition of the energy of a digraph as given by Peña and Rada [?].
Let D be a digraph on n vertices with eigenvalues $z_1, z_2, \cdots, z_n$. The energy of D is defined as

$$E(D) = \sum_{j=1}^{n} |\Re z_j|,$$

where $\Re z_j$ denotes the real part of the complex number z_j. This definition was motivated by following integral formula.
Integral expressions for for graph energy also hold for digraph energy.

Real numbers that cannot be the energy of a graph

Bapat and Pati [?] proved that the energy of a graph cannot be an odd integer. Later Pirzada and Gutman [?] proved that energy of a graph cannot be the square root of an odd integer. The following result extends these to digraphs.

Theorem L_1

Energy of a digraph cannot be of the form (i) $(2^t s)^{\frac{1}{h}}$ *with* $h \geq 1,\ 0 \leq t < h$ *and* s *odd* (ii) $(\frac{m}{n})^{\frac{1}{r}}$, *where* $\frac{m}{n}$ *is non-integral rational number and* $r \geq 1$.

The following result gives a sharp lower bound for the energy of strongly connected digraphs.

Theorem M_1

If D *is a strongly connected digraph of order* n, *then* $E(D) \geq 2$, *with equality if and only if* $D = C_r$, $r = 2, 3, 4$.

Energy of signed digraphs

A signed digraph is defined to be a pair $S = (D, \sigma)$ where $D = (V, \mathscr{A})$ is the underlying digraph and $\sigma : \mathscr{A} \to \{-1, 1\}$ is the signing function. The adjacency matrix of a signed digraph S with vertex set $\{v_1, v_2, \cdots, v_n\}$ is the $n \times n$ matrix $A(S) = (a_{ij})$, where

$$a_{ij} = \begin{cases} \sigma(v_i, v_j), & \text{if there is an arc from } v_i \text{ to } v_j, \\ 0, & \text{otherwise.} \end{cases}$$

A signed digraph is symmetric if $(u, v) \in \mathscr{A}^+(S)$ (or $\mathscr{A}^-(S)$) then $(v, u) \in \mathscr{A}^+(S)$ (or $\mathscr{A}^-(S)$), where $u, v \in V(S)$. A one to one correspondence between signed graphs and symmetric signed digraphs is given by $\Sigma \rightsquigarrow \overleftrightarrow{\Sigma}$, where $\overleftrightarrow{\Sigma}$ has the same vertex set as that of signed graph Σ and each signed edge (u, v) is replaced by a pair of symmetric arcs (u, v) and (v, u) both with same sign as that of edge (u, v). Under this correspondence a signed graph can be identified with a symmetric signed digraph. A signed digraph is said to be skew symmetric if its adjacency matrix is skew symmetric. A linear signed subdigraph of a signed digraph S is a signed subdigraph with indegree and outdegree of each vertex equal to one i.e., each component is a cycle.

The following is the coefficient theorem for signed digraphs [?].

Theorem A_2

If S is a signed digraph with characteristic polynomial

$$\phi_S(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + c_n,$$

then

$$a_j = \sum_{L \in \mathscr{L}_j} (-1)^{p(L)} \prod_{Z \in c(L)} s(Z),$$

for all $j = 1, 2, \cdots, n$, where $\mathscr{L}_j$ is the set of all linear signed subdigraphs L of S of order j, $p(L)$ denotes number of components of L and $c(L)$ denotes the set of all cycles of L and $s(Z)$ the sign of cycle Z.

The spectral criterion for cycle balance of signed digraphs given by Acharya [?] is as follows.

Theorem B_2

A signed digraph S is cycle-balanced if and only if it is cospectral with the underlying unsigned digraph.

Energy of a signed digraph.

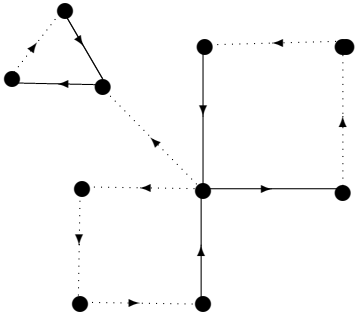
Let S be a signed digraph of order n having eigenvalues $z_1, z_2, \cdots, z_n$. The energy of S is defined as

$$E(S) = \sum_{j=1}^{n} |\Re z_j|,$$

where $\Re z_j$ denotes the real part of complex number z_j.
If S is a signed graph and $\overleftrightarrow{S}$ be its symmetric signed digraph, then clearly $A(S) = A(\overleftrightarrow{S})$ and so $E(S) = E(\overleftrightarrow{S})$. In this way, above definition generalizes the concept of energy of undirected signed graphs.(A good motivation is Coulson's integral formula).

Example 1

Let S be a signed digraph shown in Figure 3.1. Clearly, S is non cycle balanced signed digraph. By Theorem A_3, the characteristic polynomial of S is $\phi_S(x) = x^{10} + x^7 = x^7(x^3 + 1)$. The spectrum of S is $spec(S) = \{-1, 0^7, \frac{1-\sqrt{3}\iota}{2}, \frac{1+\sqrt{3}\iota}{2}\}$, where $\iota = \sqrt{-1}$, so $E(S) = 2$.



S

Figure 4.1

Example 2

Let S be an acyclic signed digraph. Then by Theorem A_3, the characteristic polynomial of S is $\phi_S(x) = x^n$, so that $spec(S) = \{0^n\}$ and hence $E(S) = 0$.

Example 3

Consider $\mathbf{S}_n$, the skew symmetric signed digraph on $n \geq 2$ vertices, then eigenvalues are of the form $\pm\iota\alpha$, where $\alpha \in \mathbb{R}$ and therefore $E(S) = 0$.

Example 4

If S is the signed directed cycle on n vertices, then the characteristic polynomial of S is $\phi_S(x) = x^n + (-1)^{[s]}$, where the symbol $[s]$ is defined as $[s] = 1$ or 0 according as S is positive or negative. If $S = C_n$, then $spec(S) = \{e^{\frac{2\iota j\pi}{n}},\ j = 0, 1, \cdots, n-1\}$ so that $E(S) = \sum_{j=0}^{n-1} |\cos(\frac{2j\pi}{n})|$. If $S = \mathbf{C}_n$, then $spec(S) = \{e^{\frac{\iota(2j+1)\pi}{n}},\ j = 0, 1, \cdots, n-1\}$ so that $E(S) = \sum_{j=0}^{n-1} |\cos(\frac{(2j+1)\pi}{n})|$. In particular if $S = \mathbf{C}_4$, then $spec(S) = \{\frac{1-\iota}{\sqrt{2}}, \frac{1+\iota}{\sqrt{2}}, \frac{-1-\iota}{\sqrt{2}}, \frac{-1+\iota}{\sqrt{2}}\}$ and $E(S) = 2\sqrt{2}$.

Now we have the following result.

Theorem C$_2$

Let S be a signed digraph on n vertices and $S_1, S_2, \cdots, S_k$ be its strong components. Then

$$E(S) = \sum_{j=1}^{k} E(S_j).$$

As in signed graphs, we denote positive and negative cycles of order n by C_n and $\mathbf{C}_n$ respectively. The following are exact formulae for the energy of signed directed cycles.

$$E(C_n) = \begin{cases} 2\cot\frac{\pi}{n}, & \text{if } n \equiv 0 \pmod 4, \\ 2\csc\frac{\pi}{n}, & \text{if } n \equiv 2 \pmod 4, \\ \csc\frac{\pi}{2n}, & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

and

$$E(\mathbf{C_n}) = \begin{cases} 2\csc\frac{\pi}{n}, & \text{if } n \equiv 0 \pmod 4, \\ 2\cot\frac{\pi}{n}, & \text{if } n \equiv 2 \pmod 4, \\ \csc\frac{\pi}{2n}, & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

Recall a signed digraph of order n is said to be unicyclic if it has n arcs and a unique cycle of length $r \leq n$. Following result characterizes unicyclic signed digraphs with minimal and maximal energy.

Theorem D_2

Energy of positive (negative) cycles increases monotonically with respect to the order. Among all cycle-balanced (non-cycle-balanced) unicyclic signed digraphs on n vertices, the cycle has the largest energy. Moreover, minimal energy is attained in unicyclic signed digraph with unique cycle C_r, where $r = 2$ or 3 or 4 ($\mathbf{C}_2$).

Upper bounds for the energy of signed digraphs.

Let S be a signed digraph of order n with adjacency matrix $A(S) = (a_{ij})$. The powers of $A(S)$ count the number of walks in signed manner. Let $w_{ij}^{+}(l)$ and $w_{ij}^{-}(l)$ respectively denote the number of positive and negative walks of length l from v_i to v_j. The following result relates the integral powers of the adjacency matrix with the number of positive and negative walks.

Theorem E$_2$

If A is an adjacency matrix of a signed digraph on n vertices, then $[A^l]_{ij} = w_{ij}^{+}(l) - w_{ij}^{-}(l)$.

In the signed digraph S, let c_m^{+} denote the number of positive closed walks of length m and c_m^{-} the number of negative closed walks of length m. In view of the fact that sum of eigenvalues of a matrix equals to its trace, we have the following observation.

Corollary 2.1

If $z_1, z_2, \cdots, z_n$ are the eigenvalues of a signed digraph S, then $\sum_{j=1}^{n} z_j^m = c_m^{+} - c_m^{-}$.

Lemma 2.1

Let S be a signed digraph having n vertices and a arcs and let $z_1, z_2, \cdots, z_n$ be its eigenvalues. Then (i) $\sum_{j=1}^{n}(\Re z_j)^2 - \sum_{j=1}^{n}(\Im z_j)^2 = c_2^+ - c_2^-$, (ii) $\sum_{j=1}^{n}(\Re z_j)^2 + \sum_{j=1}^{n}(\Im z_j)^2 \leq a = a^+ + a^-$.

Theorem F_2

Let S be a signed digraph with n vertices and $a = a^+ + a^-$ arcs, and let $z_1, z_2, \cdots, z_n$ be its eigenvalues. Then $E(S) \leq \sqrt{\frac{1}{2}n(a + c_2^+ - c_2^-)}$.

Remark 2.1

(i). The upper bound in Theorem I_3 is attained by signed digraphs $S_1 = (\frac{n}{2}\overleftrightarrow{K_2}, +)$, $S_2 = (\frac{n}{2}\overleftrightarrow{K_2}, -)$, (where $(\overleftrightarrow{K}_2, +)$ and $(\overleftrightarrow{K}_2, -)$ respectively denote symmetric digraphs obtained from $+K_2$ and $-K_2$) and skew symmetric signed digraph of order n. Note that $spec(S_1) = spec(S_2) = \{-1^{(\frac{n}{2})}, +1^{(\frac{n}{2})}\}$ and eigenvalues of skew symmetric signed digraph of order n are of the form $\pm\iota\alpha$, where $\alpha \in \mathbb{R}$.
(ii). The Above result extends McClleland's inequality for signed graphs [?] which states that $E(S) \leq \sqrt{2pq}$, holds for every signed graph with p vertices and q edges. Let $\overleftrightarrow{S}$ be the symmetric signed digraph of signed graph S, then in $\overleftrightarrow{S}$, $a = 2q = c_2^+ = c_2^+ - c_2^-$. By Theorem I_3, $E(S) = E(\overleftrightarrow{S}) \leq \sqrt{\frac{1}{2}p(2q+2q)} = \sqrt{2pq}$.

The following result gives the sharp upper bound of energy of signed digraphs in terms of the number of arcs.

Theorem G_2

Let S be a signed digraph with a arcs. Then $E(S) \leq a$ with equality if and only if $S = (\frac{a}{2}\overleftrightarrow{K_2}, +)$ or $S = (\frac{a}{2}\overleftrightarrow{K_2}, -)$ plus some isolated vertices.

Equienegrgetic signed digraphs.

Two signed digraphs are said to be isomorphic if their underlying digraphs are isomorphic such that the signs are preserved. Any two isomorphic signed digraphs are obviously cospectral.
Two nonisomorphic signed digraphs S_1 and S_2 of same order are said to be equienergetic if $E(S_1) = E(S_2)$. Rada [?] proved the existence of pairs of non-symmetric and non cospectral equienergetic digraphs. Cospectral signed digraphs are obviously equienergetic, therefore the problem of equienergetic signed digraphs reduces to the problem of construction of non cospectral pairs of equienergetic signed digraphs such that for every pair not both signed digraphs are cycle balanced.
We have the following result.

Theorem H_2

Let S be a signed digraph of order n having eigenvalues $z_1, z_2, \cdots, z_n$ such that $|\Re z_j| \leq 1$ for every $j = 1, 2, \cdots, n$. Then $E(S \times \overleftrightarrow{K_2}) = 2n$.

Now we have the following consequence.

Corollary 2.2

For $n \geq 2$, $E(\mathbf{C}_n \times \overleftrightarrow{K_2}) = E(C_n \times \overleftrightarrow{K_2}) = 2n$. Moreover, $\mathbf{C}_n \times \overleftrightarrow{K_2}$ and $C_n \times \overleftrightarrow{K_2}$ are non cospectral signed digraphs with $2n$ vertices.

Example 5

For each odd n, $\mathbf{C}_n$ and C_n is a non cospectral pair of equienergetic signed digraphs, because $spec(\mathbf{C}_n) = -spec(C_n)$ and $1 \notin spec(\mathbf{C}_n)$ but $1 \in spec(C_n)$.

From Corollary 3.2 and Example 7, we see for each positive integer $n \geq 3$, there exits a pair of non cospectral signed digraphs with one signed digraph cycle balanced and another non cycle balanced. Now we construct pairs of non cospectral equienergetic signed digraphs of order $2n$, $n \geq 5$ with both constituents non cycle balanced. Let P_n^l $(n \geq l+1)$ be a signed digraph obtained by identifying one pendant vertex of the path P_{n-l+1} with any vertex of $\mathbf{C}_l$. Sign of non cyclic arcs is immaterial.

Theorem I_2

For each $n \geq 5$, $P_n^3 \times \overleftrightarrow{K_2}$ and $P_n^4 \times \overleftrightarrow{K_2}$ is a pair of non cospectral equienergetic signed digraphs of order and energy equal to $2n$.

Bapat and Pati [?] proved that the energy of a graph cannot be an odd integer. Pirzada and Gutman [?] proved that energy of a graph cannot be the square root of an odd integer. In Chapter 1 we saw that these results hold good for digraphs. The next result extends these results to signed digraphs.

Theorem J_2

Energy of a digraph cannot be of the form (i) $(2^t s)^{\frac{1}{h}}$ with $h \geq 1$, $0 \leq t < h$ and s odd (ii) $(\frac{m}{n})^{\frac{1}{r}}$, where $\frac{m}{n}$ is non-integral rational number and $r \geq 1$.

Spectra and energy of bipartite signed digraphs.

We next state Coulson's integral formula for the energy of signed digraphs.

Theorem A$_3$

Let S be a signed digraph with n vertices having characteristic polynomial $\phi_S(x)$. Then

$$E(S) = \sum_{j=1}^{n} |\Re z_j| = \frac{1}{\pi} \int_{-\infty}^{\infty} (n - \frac{\iota x \phi'_S(\iota x)}{\phi_S(\iota x)}) dx,$$

where $z_1, z_2, \cdots, z_n$ are the eigenvalues of signed digraph S and $\int_{-\infty}^{\infty} F(x)dx$ denotes principle value of the respective integral.

Theorem B$_3$

If S is a signed digraph on n vertices, then

$$E(S) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{x^2} \log |x^n \phi_S(\frac{\iota}{x})| dx.$$

Theorem C$_3$

If S is a signed digraph on n vertices with characteristic polynomial $\phi_S(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$, then

$$E(S) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{x^2}\log[(\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^j a_{2j}x^{2j})^2 + (\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^j a_{2j+1}x^{2j+1})^2]dx.$$

Recall a sidigraph S is bipartite if its underlying digraph is bipartite. The following result by Esser and Harary [**?**] characterizes strongly connected bipartite digraphs in terms of spectra.

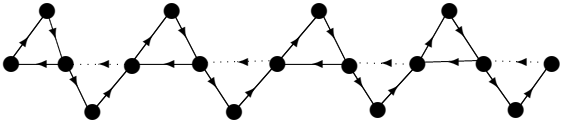
Theorem D$_3$

A strongly connected digraph D is bipartite if and only if its spectrum is invariant under multiplication by -1.

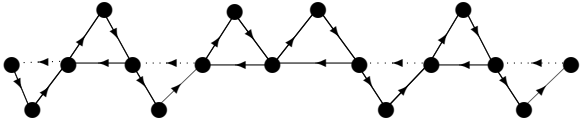
Two signed digraphs S_1 and S_2 are said to have the pairing property if z is an eigenvalue of S_1, then $-z$ is an eigenvalue of S_2 and vice versa. In case $S_1 = S_2 = S$ (say), then we say S has the pairing property. Let S be a bipartite signed digraph the characteristic polynomial of S is given by $\phi_S(z) = z^\delta\psi(z^2)$, where δ is a nonnegative integer and $\psi(z^2)$ is a polynomial in z^2. Therefore the spectrum of a bipartite signed digraph remains invariant under multiplication by -1 i.e., S has the pairing property.

Remark 3.1

Unlike in digraphs, the converse of Theorem D_4 is not true for signed digraphs. For example signed digraphs S_1 and S_2 in Fig. 4.1 are two strongly connected non-bipartite signed digraphs of order 17. It is easy to check that $\phi_{S_1}(x) = \phi_{-S_1}(x) = x^{17} + 3x^{11} + x^5$ and $\phi_{S_2}(x) = \phi_{-S_2}(x) = x^{17} + x^{11} + x^5$.



S_1



S_2

Fig. 4.2 A pair of non-bipartite sidigraphs having pairing property.

As in bipartite graphs or signed graphs, in general, the even coefficients of non-cycle-balanced bipartite signed digraphs does not alternate in sign. For example, the characteristic polynomials of non-cycle balanced bipartite sidigraphs S and T in Fig. 4.3 are $\phi_S(x) = x^4 - x^2 - 1$ and $\phi_T(x) = x^4 + x^2$. Clearly, even coefficients do not alternate in sign. Now, consider the non-cycle-balanced bipartite signed digraph S_1 in Fig. 2, the characteristic polynomial is $\phi_{S_1}(x) = x^6 - x^4 + 2x^2$. In this case, even coefficients alternate in sign.

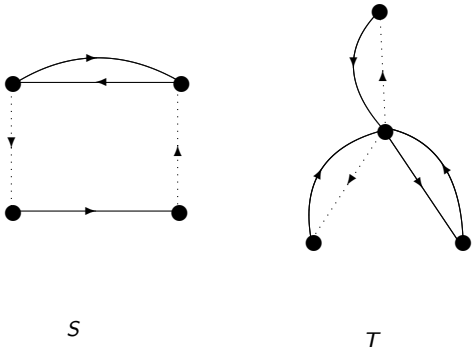


Fig. 4.3 Bipartite sidigraphs with non-alternating coefficients

A natural question arises which bipartite signed digraphs have alternating even coefficients. In this regard, We show bipartite signed digraphs on n vertices with each cycle of length $\equiv 0 \pmod 4$ negative (i.e., containing an odd number of negative arcs) and each cycle of length $\equiv 2 \pmod 4$ positive (i.e., containing an even number of negative arcs) has characteristic polynomial with alternating even coefficients. We denote this class of bipartite signed digraphs by Δ_n^1.

We also study another class of bipartite signed digraphs on n vertices with all cycles negative (i.e., each cycle has an odd number of negative arcs) and show a signed digraph in this class has characteristic polynomial with all nonnegative coefficients. We denote this class of bipartite signed digraphs by Δ_n^2.

Theorem E3

If $S \in \Delta_n^1$, then

$$\phi_S(x) = x^n + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^j c_{2j}(S) x^{n-2j}, \tag{3}$$

where $c_{2j}(S) = |\$_{2j}|$ is the cardinality of the set $\$_{2j}$.

Remark 3.2

Here we note that there exist bipartite and non-bipartite non-cycle-balanced signed digraphs not in Δ_n^1 which have characteristic polynomial with alternating coefficients. Signed digraphs S_1 and S_2 in Fig. 4.3 clearly does not belong to Δ_n^1. By coefficient Theorem, $\phi_{S_1}(x) = x^6 - x^4 + 2x^2$ and $\phi_{S_2}(x) = x^6 - 1$.

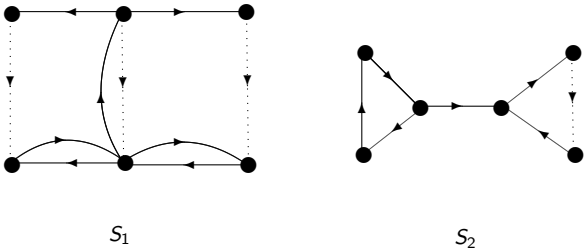


Fig. 4.4 A pair of signed digraphs not in Δ_n^1 but having alternating coefficients.

The following result shows that the characteristic polynomial of a sidigraph in Δ_n^2 is of the form (2). Proof is same as the proof of Theorem E_4.

Theorem F_3

Let $S \in \Delta_n^2$. Then the characteristic polynomial is given by

$$\phi_S(z) = z^n + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} c_{2j}(S) z^{n-2j},$$

where $c_{2j}(S) = |\$_{2j}|$ is the cardinality of the set $\$_{2j}$.

Corollary 3.1

Let $S_1 \in \Delta_n^1$ and $S_2 \in \Delta_n^2$ have same underlying digraph D. Then $spec(S_1) = \iota spec(S_2)$.

Given signed digraphs S_1 and S_2 in Δ_n^1, by Theorem E_4, for $i = 1, 2$, we have

$$\phi_{S_i(x)} = x^n + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^j c_{2j}(S_i) x^{n-2j},$$

where $c_{2j}(S_i)$ are non negative integers for all $j = 1, 2, \cdots, \lfloor \frac{n}{2} \rfloor$. If $c_{2j}(S_1) \leq c_{2j}(S_2)$ for all $j = 1, 2, \cdots, \lfloor \frac{n}{2} \rfloor$, then we define $S_1 \preceq S_2$. If in addition $c_{2j}(S_1) < c_{2j}(S_2)$ for some $j = 1, 2, \cdots, \lfloor \frac{n}{2} \rfloor$, then we write $S_1 \prec S_2$. Clearly $\preceq$ is a quasi-order relation (i.e., a reflexive and transitive relation). The following result shows that energy increases in Δ_n^1 with respect to this quasi-order relation.

Theorem G3

If $S \in \Delta_n^1$, then

$$E(S) = \frac{2}{\pi} \int_0^\infty \frac{1}{x^2} \log[1 + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} c_{2j}(S) x^{2j}] dx.$$

In particular, if $S_1, S_2 \in \Delta_n^1$ and $S_1 \prec S_2$ then $E(S_1) < E(S_2)$.

Problem of decrease in energy by deleting an edge in weighted graph or an arc form cycle of length 2 of a bipartite digraph has been studied in [?, ?]. As in digraphs, in general it is not possible to predict the change in the energy of a non-cycle-balanced signed digraph by deleting an arc from a cycle of length 2. It can decrease, increase or remain same by deleting an arc of a cycle of length 2 as can be seen in the following example.

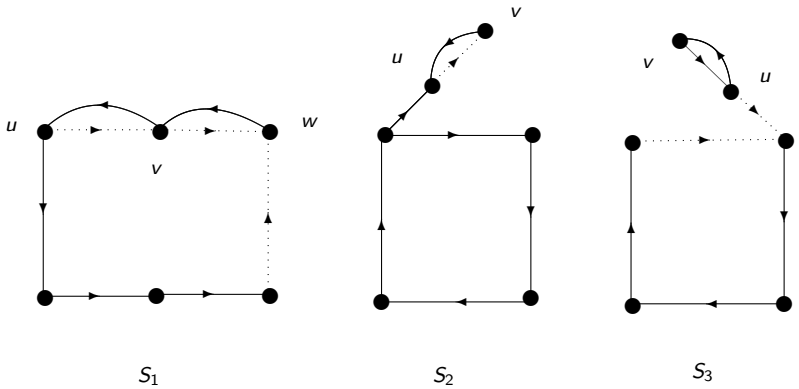


Fig. 4.5 Arc deletion and energy change for signed digraphs in Δ_n^1.

Example 6

Consider the sidigraphs S_1, S_2 and S_3 as shown in Fig. 4. It is easy to see that $\phi_{S_1}(x) = x^6 + 2x^4 + 1$ and $\phi_{S_1^{(u,v)}}(x) = x^6 + x^4 + 1$, where $S_1^{(u,v)}$ denotes the sidigraph obtained by deleting the arc (u, v). Note $E(S_1) \approx 2.4916$ and $E(S_1^{(u,v)}) \approx 2.9104$. So the energy increases in this case. Also, $\phi_{S_2}(x) = x^6 + x^4 - x^2 - 1$ and $spec(S_2) = \{-1, 1, -\iota^{(2)}, \iota^{(2)}\}$ so that $E(S_2) = 2$. If we delete arc (u, v), the resulting sidigraph has eigenvalues $\{-1, 0^{(2)}, 1, -\iota, \iota\}$ so the energy of the resulting sidigraph is again 2. That is, the energy remains same in this case. It is not difficult to check that $E(S_3) = 2 + 2\sqrt{2}$ and $E(S_3^{(u,v)}) = 2\sqrt{2}$. So the energy decreases in this case.

The following result as an application of Theorem C_4 and E_4 shows that the energy of a sidigraph in Δ_n^1 decreases when we delete an arc from a cycle of length 2.

Theorem H_3

Let S be a sidigraph in Δ_n^1 with a pair of symmetric arcs and let S' be the sidigraph obtained by deleting one of these arcs. Then $E(S') < E(S)$.

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THANK YOU