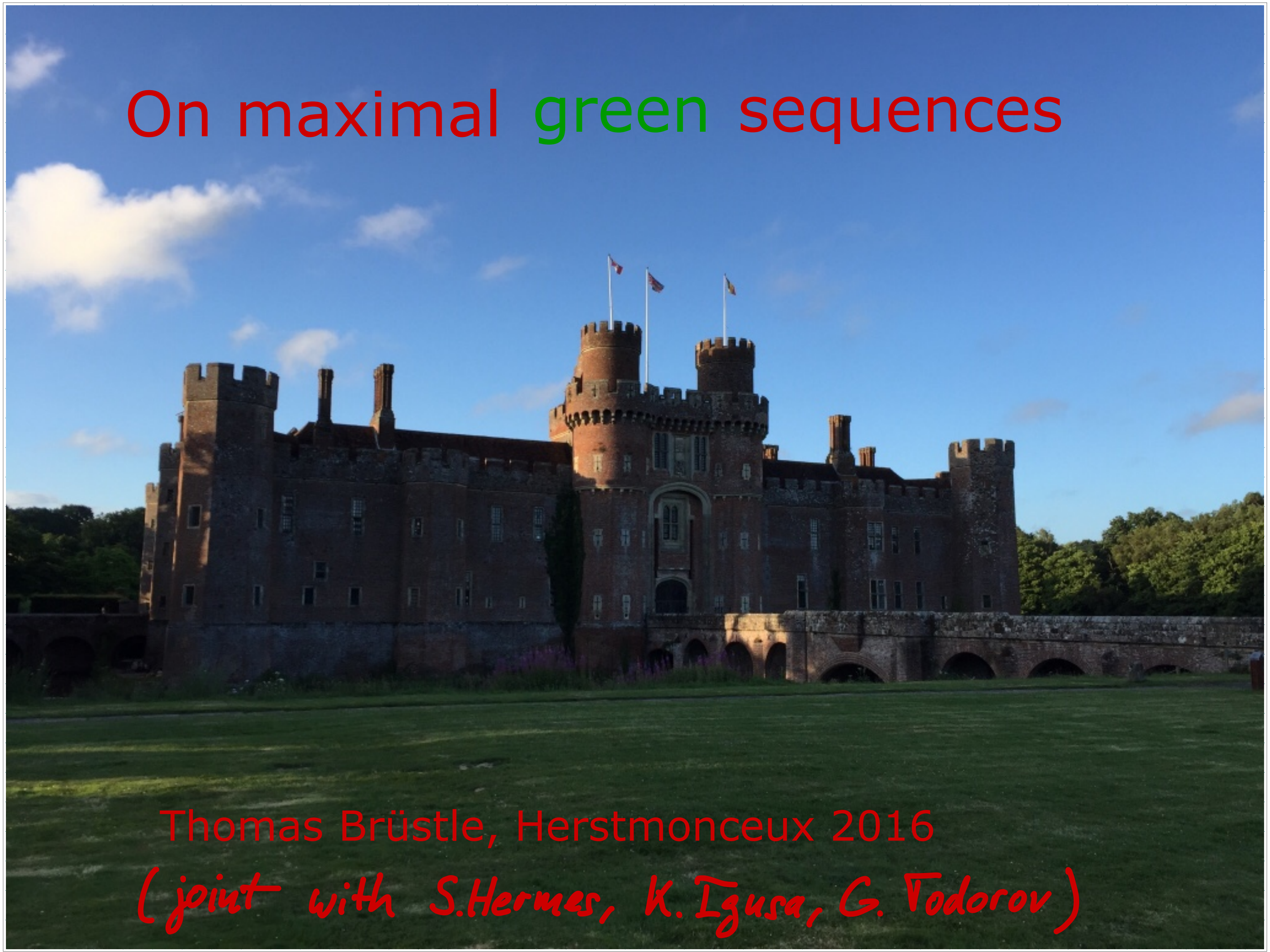
On maximal green sequences

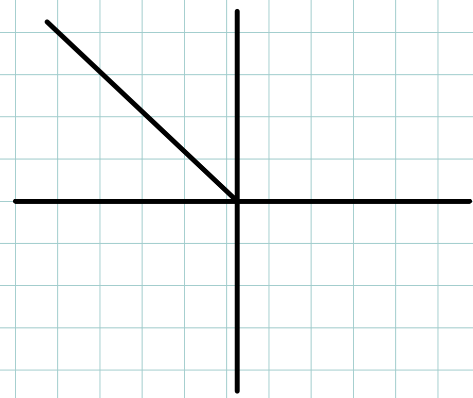
Thomas Brüstle, Herstmonceux 2016

(joint with S. Hermes, K. Igusa, G. Todorov)

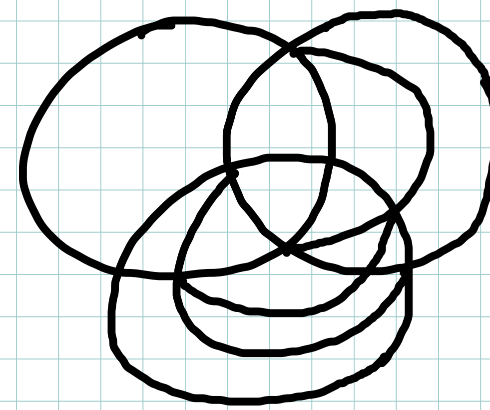


1) Setup: Stability Conditions

A_2



A_3



A. King: Q quiver, n vertices

$\theta : \mathbb{Z}^n \longrightarrow \mathbb{R}$ additive

$$\mathbb{Z}^n \cong K_0(\operatorname{rep} Q)$$
$$\cup$$
$$\{\text{positive roots}\}$$

Q quiver, n vertices

$$q : \mathbb{Z}^n \longrightarrow \mathbb{Z}, \; x \longmapsto \sum_{i=1}^{n} x_i^2 - \sum_{i \xrightarrow{\alpha} j} x_i x_j$$

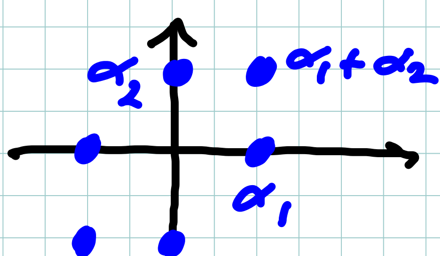
quadratic form

q positive definite $\Longleftrightarrow$ Q Dynkin

$M \in \mathbb{Z}^n$ real root $\Longleftrightarrow$ $q(M) = 1$

$Q = \overset{1}{\circ} \longrightarrow \overset{2}{\circ}$ type A_2, $q(x) = x_1^2 + x_2^2 - x_1 x_2$

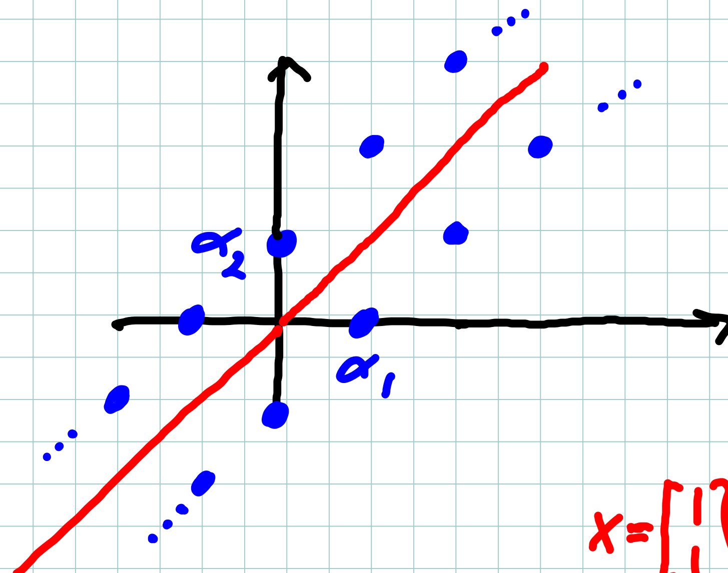
positive real roots are $\alpha_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\alpha_1 + \alpha_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



$M \in \mathbb{Z}^n$ real root $\iff$ $q(M) = 1$

$Q = \overset{1}{\circ} \rightrightarrows \overset{2}{\circ}$ affine type A, $q(x) = x_1^2 + x_2^2 - 2x_1x_2$

real roots are $\left\{ \begin{bmatrix} n \\ n+1 \end{bmatrix}, \begin{bmatrix} n+1 \\ n \end{bmatrix} : n \in \mathbb{Z} \right\}$



$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow q(x) = 0$

"imaginary root"

A. King: Q quiver, n vertices

$$\theta : \mathbb{Z}^n \longrightarrow \mathbb{R} \text{ additive}$$

$$M \in \mathbb{Z}^n \text{ positive root}$$

M θ-semi-stable if $\theta(M) = 0$ and $\theta(N) \geq 0$

for all N positive root, subrepresentation of M

A. King: Q quiver, n vertices

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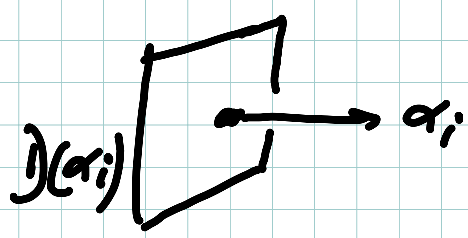
$\mathbb{R}^n \cong \{\theta_y \mid y \in \mathbb{R}^n\}$ $\quad \theta_y : \mathbb{R}^n \to \mathbb{R},\ x \mapsto (x,y)$

space of stability functions

For M pos. real root, $D(M) = \{\theta_y : M \text{ is } \theta_y\text{-semi-stable}\}$

$M = \alpha_i$ simple root : $\nexists\, N \subset M$

$M = \alpha_i$ θ_y-stable $\Leftrightarrow$ $(y, \alpha_i) = 0$



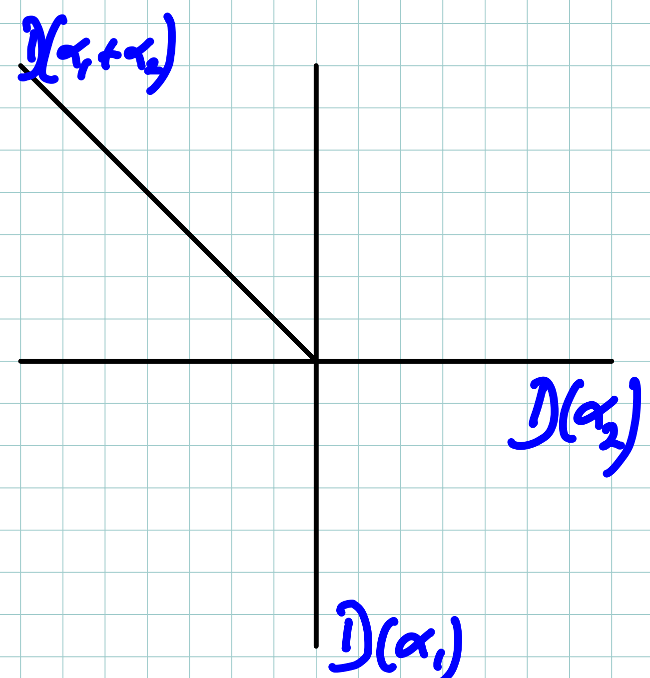
example: $Q = \overset{1}{\circ} \longrightarrow \overset{2}{\circ}$ $\quad A_2$

$M = \alpha_1 + \alpha_2$ pos. real root : $M = \begin{matrix} \alpha_1 \\ \downarrow \\ \alpha_2 \end{matrix}$ $\begin{matrix} \alpha_1 \text{ quotient} \\ \\ \alpha_2 \text{ submodule} \end{matrix}$

$(y, M) = 0 \iff y_1 + y_2 = 0$

$D(M) = \{\theta_y : y_1 + y_2 = 0, \ y_2 \geq 0\}$

Space of stability functions:

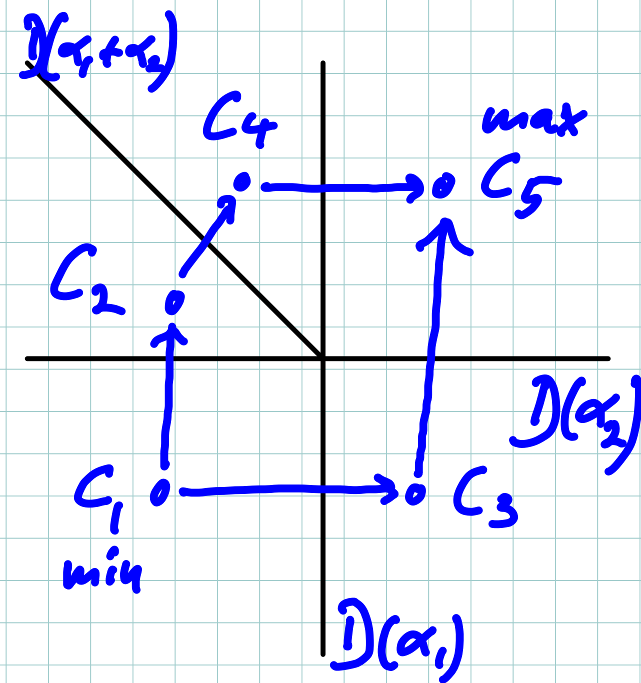


Chambers = components of

$$\mathbb{R}^n \setminus \bigcup_{M \text{ pos. real root}} D(M)$$

example: $Q = \overset{1}{\circ} \longrightarrow \overset{2}{\circ}$ $\quad A_2$

Space of stability functions:



$$C_{min} = \{ Y \in \mathbb{R}^n : (Y, \alpha_i) < 0 \quad \forall i = 1, \ldots, n \}$$

$$C_{max} = \{ Y \in \mathbb{R}^n : (Y, \alpha_i) > 0 \quad \forall i = 1, \ldots, n \}$$

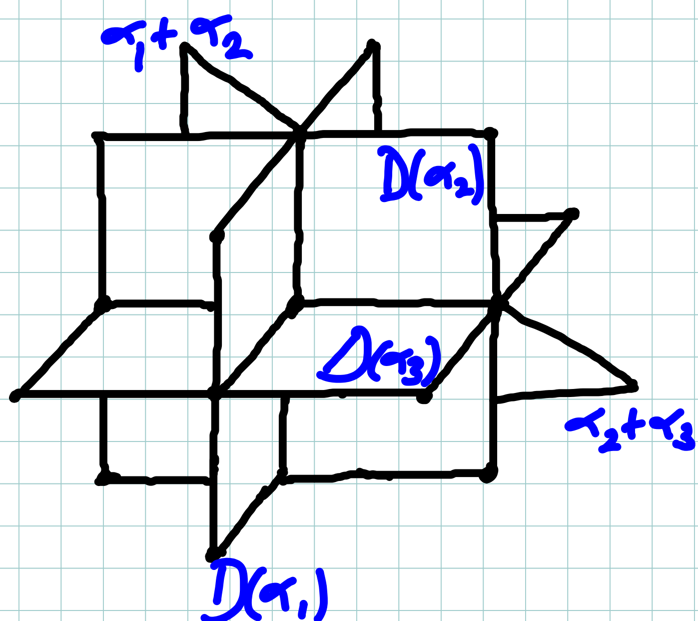
MGS = maximal green sequence

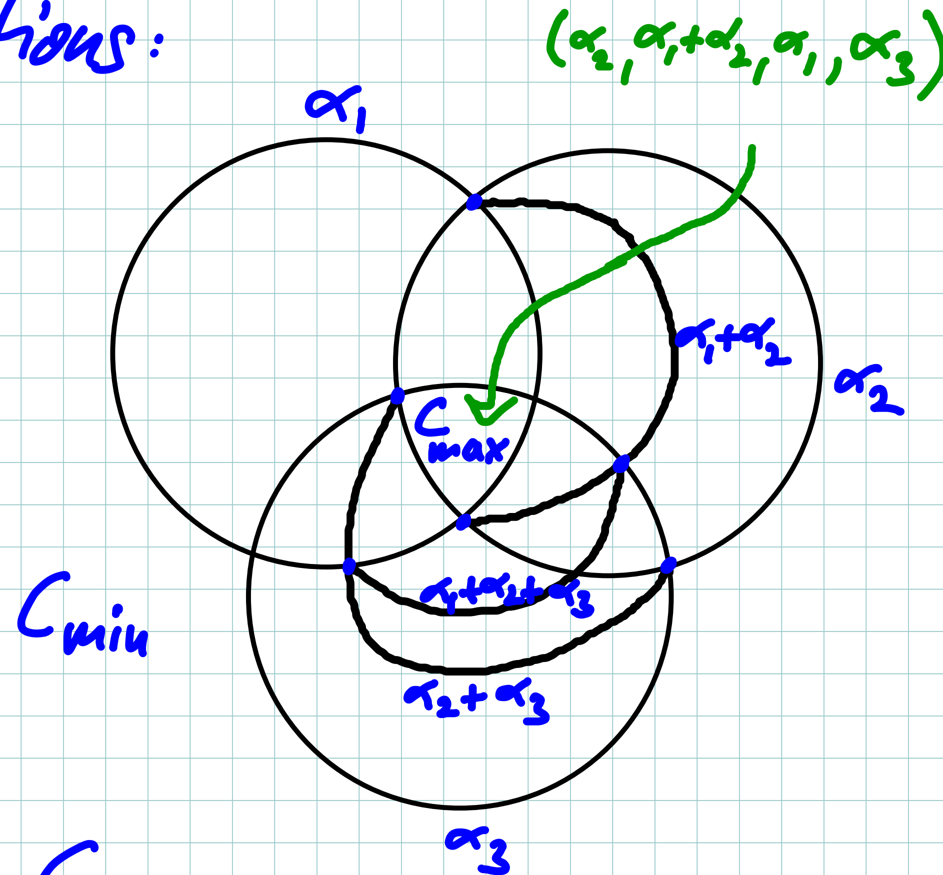
= path from C_{min} to C_{max} passing through walls $D(M)$ separately and in positive direction: $(Y, M) < 0 \rightsquigarrow (Y, M) > 0$

A_2 has 2 MGS: $(\alpha_2, \alpha_1+\alpha_2, \alpha_1)$ and (α_1, α_2)

example: $Q = \overset{1}{\circ} \longrightarrow \overset{2}{\circ} \longrightarrow \overset{3}{\circ}$ $\quad A_3$

Space of stability functions:





$14 = \mathrm{Cat}(4)$ chambers C_i

9 MGS

lengths $3 \le \ell \le 6$

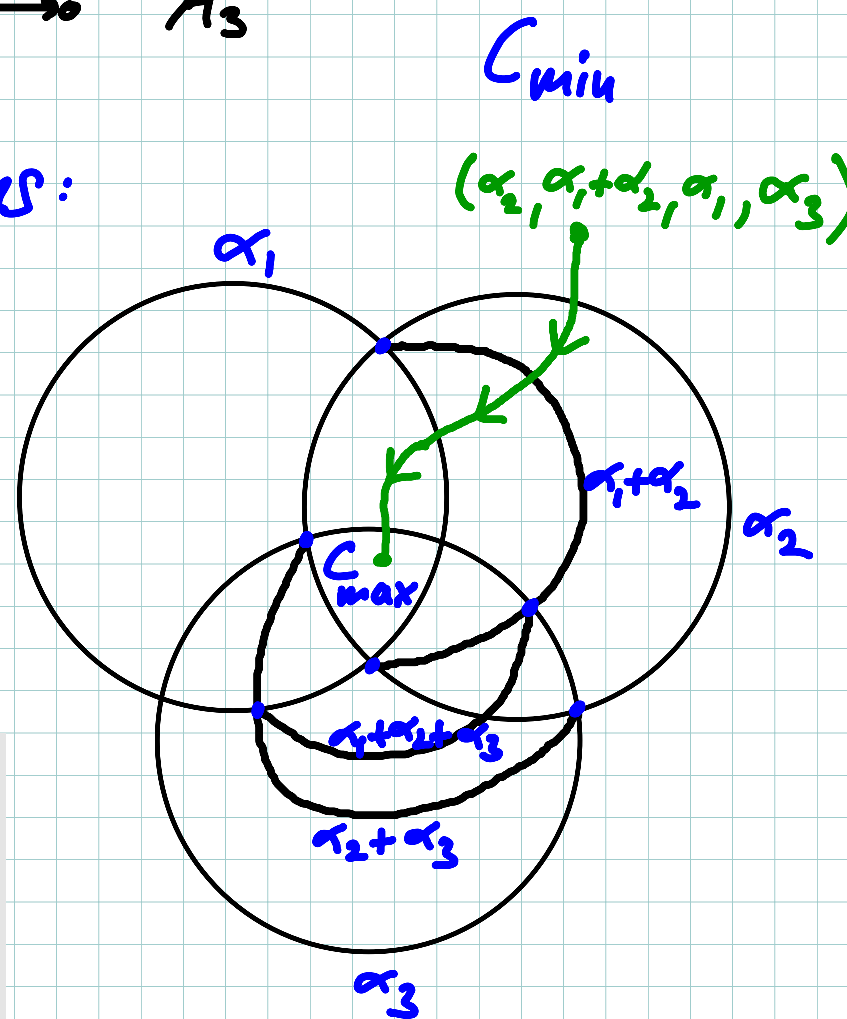
example: $Q = \overset{1}{\circ} \longrightarrow \overset{2}{\circ} \longrightarrow \overset{3}{\circ}$ A_3

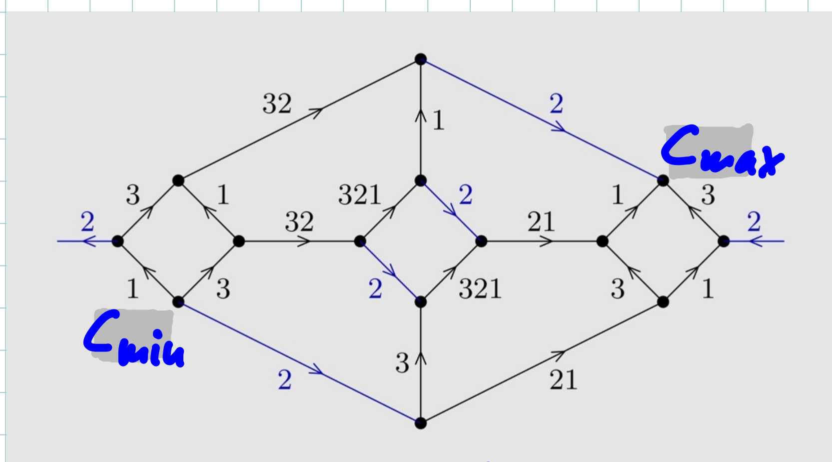
Space of stability functions:

$14 = Cat(4)$ chambers

9 MGS

lengths $3 \leq \ell \leq 6$





MGS = maximal chain

2) Basic Questions on MGS:

a) Existence, Finiteness

b) (per) mutations

c) Relevance

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c) Relevance

c.1) Reineke, Keller:

$\{\text{MGS's for } Q\} \longmapsto$ Quantum dilogarithm identities

$Q = \overset{1}{\circ} \longrightarrow \overset{2}{\circ}$ 2 MGS (α_1, α_2) and $(\alpha_2, \alpha_1+\alpha_2, \alpha_1)$ yield:

Theorem 1.2 (Schützenberger [62], Faddeev-Volkov [16], Faddeev-Kashaev [17]). *For two indeterminates y_1 and y_2 which q-commute in the sense that*

$$y_1 y_2 = q y_2 y_1 ,$$

we have the equality

$$\mathbb{E}(y_1)\mathbb{E}(y_2) = \mathbb{E}(y_2)\mathbb{E}(q^{-1/2} y_1 y_2)\mathbb{E}(y_1). \tag{1.3}$$

As shown in [17], this equality implies the classical 'pentagon identity' for Rogers' dilogarithm.

2) Basic Questions on MGS:
c) Relevance

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c.2) Quantum Field Theory
[Alim, Cecotti; Córdova, Espahbodi; Rastogi, Vafa, ...]

MGS $\longmapsto$ Spectrum of BPS particle

Show existence of MGS when Q is given by triangulation of a (non-closed) surface

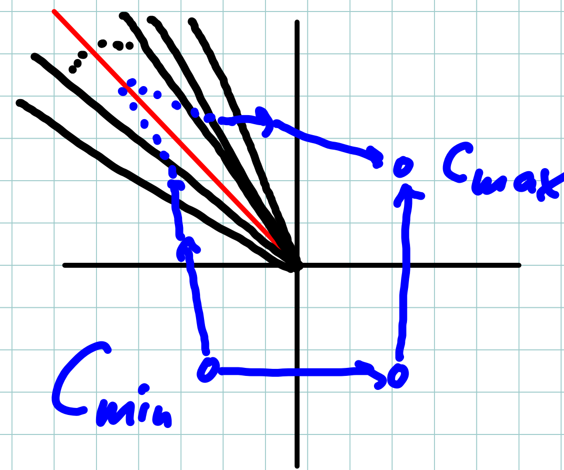
2) Basic Questions on MGS:

a) Existence, Finiteness

$Q = 1 \rightrightarrows 2$

space of stability functions:

1 MGS (α_1, α_2)



$$Q = \begin{tikzcd} & 3 \arrow[dl] & \\ 1 \arrow[rr, shift left] \arrow[rr, shift right] & & 2 \arrow[ul] \end{tikzcd}$$

stability space

$0 < \# MGS < \infty$

$$Q = \begin{tikzcd} & 3 \arrow[dl, shift left] \arrow[dl, shift right] & \\ 1 \arrow[rr, shift left] \arrow[rr, shift right] & & 2 \arrow[ul, shift left] \arrow[ul, shift right] \end{tikzcd}$$

stability space

0 MGS

2) Basic Questions on MGS:

a) Existence, Finiteness

<u>Theorem:</u> [BHIT]

Q mutation-equivalent to affine Dynkin quiver

Then $0 < \# MGS < \infty$

2) Basic Questions on MGS:

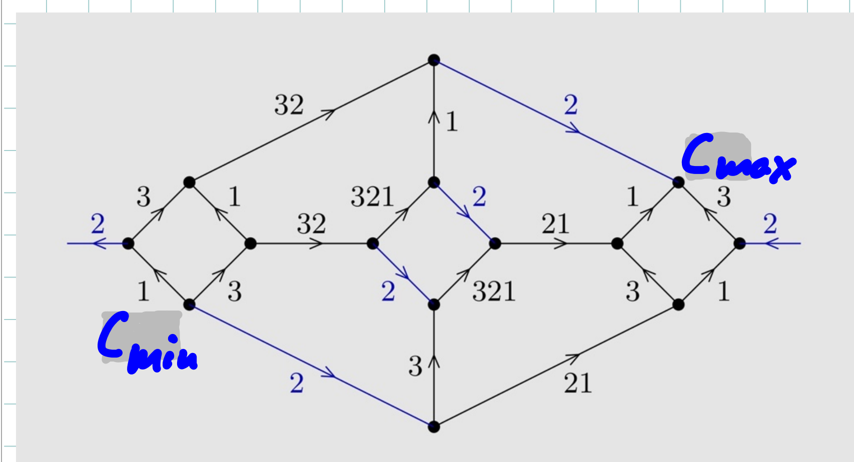
b) (per)mutations

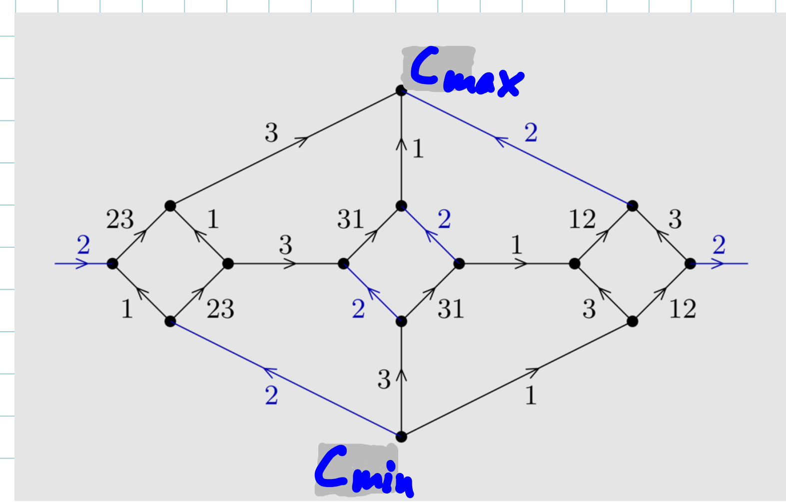
example:

$$Q = \overset{1}{\circ} \longrightarrow \overset{2}{\circ} \longrightarrow \overset{3}{\circ} \quad \xmapsto[\mu_2]{\text{mutation}} \quad \mu_2(Q) = \begin{tikzcd} & \overset{2}{\circ} \arrow[dl] & \\ \underset{1}{\circ} \arrow[rr] & & \underset{3}{\circ} \arrow[ul] \end{tikzcd}$$

9 MGS
lengths $3 \le \ell \le 6$

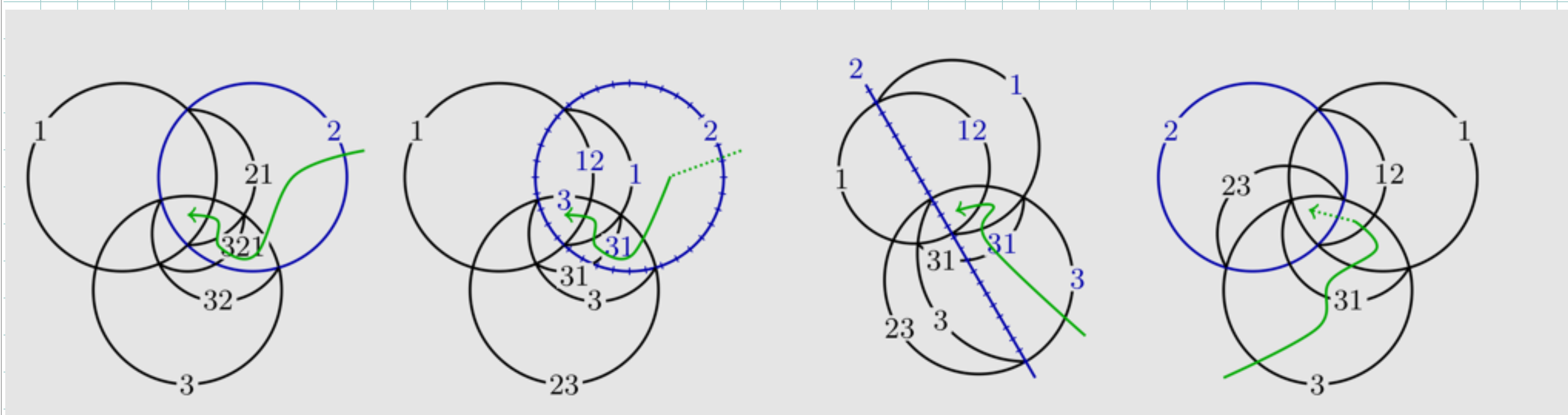
9 MGS
lengths $4 \le \ell \le 5$





example:

$$Q = \overset{1}{\circ} \longrightarrow \overset{2}{\circ} \longrightarrow \overset{3}{\circ} \quad \xmapsto[\mu_2]{\text{mutation}} \quad \mu_2(Q) = \begin{tikzcd} & \overset{2}{\circ} \arrow[dl] & \\ \underset{1}{\circ} \arrow[rr] & & \underset{3}{\circ} \arrow[ul] \end{tikzcd}$$



$(2,3,321,21,1) \longmapsto (3,31,1,12,2)$ c-vectors

$(2,3,1,3,1) \longmapsto (3,1,3,1,2)$ indices

2) Basic Questions on MGS:
b) (per)mutations

Theorem: [BHIT] (Rotation Lemma)

Let $(r_0, r_1, \ldots, r_m)$ be a MGS for Q, $1 \leq r_i \leq n$
Consider mutation $\mu_{r_0} : Q \longmapsto \mu_{r_0}(Q)$.
Then $(r_1, r_2, \ldots, r_m, r_{m+1})$ is MGS for $\mu_{r_0}(Q)$,
for a unique index r_{m+1}.