Bi-Cayley graphs

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Group Actions & Algebraic Combinatorics
(Herstmonceux, 11-15 July 2016)

Graph symmetries

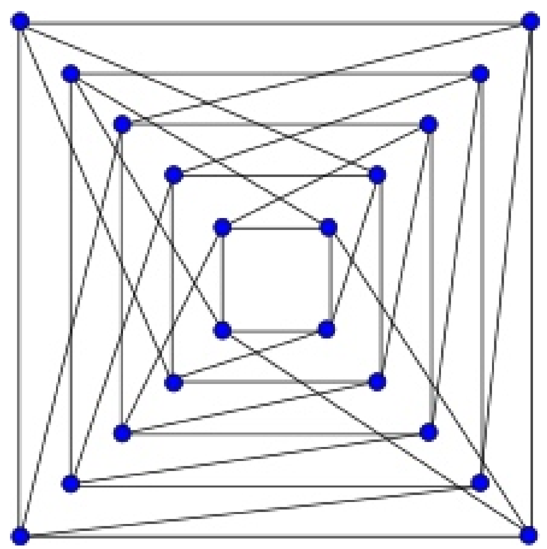
A graph is called vertex-transitive if its automorphism group A has a single orbit on vertices, or edge-transitive if A has a single orbit on edges, or arc-transitive (or symmetric) if A has a single orbit on the set of arcs — where an arc is an ordered pair (v, w) of adjacent vertices.

A graph that is regular and edge-transitive but not vertex-transitive is semisymmetric. A graph that is vertex- and edge-transitive but not arc-transitive is half-arc-transitive. Semisymmetric graphs are bipartite, and half-arc-transitive graphs are regular with even valency.

An s-arc is a path of length s in which any three consecutive vertices are distinct, and a graph is s-arc-transitive if its automorphism group A has a single orbit on s-arcs.

Semisymmetric graphs

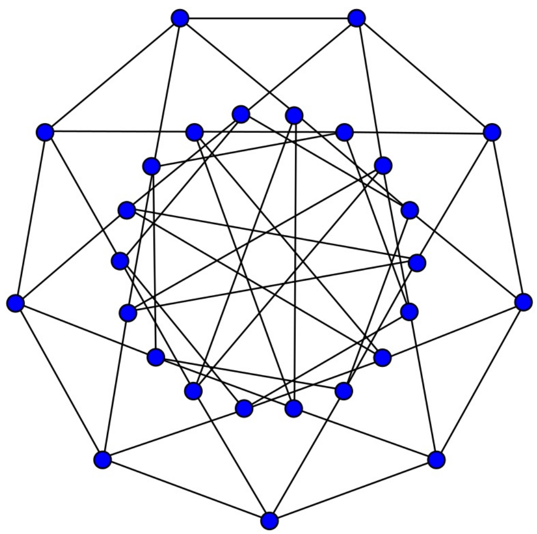
These are edge-transitive & regular but not vertex-transitive. As such, they are all bipartite, with the two parts being the orbits of the automorphism group on orbits. The smallest example is the 4-valent Folkman graph on 20 vertices:



Half-arc-transitive graphs

These are vertex- and edge- but not arc-transitive. As such, they all have even valency > 2. (Cycles are arc-transitive.)

The smallest example is the 4-valent Holt graph (now aka the Doyle graph), on 27 vertices:



Cayley graphs

A graph X admitting a group G of automorphisms that acts regularly (sharply-transitively) on vertices is a Cayley graph.

If we fix a vertex v, and then let S be the subset of G consisting of automorphisms that take v to a neighbour, then we denote X by Cay(G, S), and call X a Cayley graph for G relative to S. This gives another way to define it:

For the vertices of X take the elements of G, and for the edges take all pairs $\{g, sg\}$ where $s \in S$ and $g \in G$. Then G acts as a group of automorphisms by right multiplication.

Examples

- The cycle graph C_n is VT, ET and AT, and Cayley
- The complete graph K_n is VT, ET and AT, and Cayley: $K_n = \text{Cay}(G, G \setminus \{1\})$ for every group G of order n.
- The complete bipartite graph $K_{n,n}$ is VT, ET and AT, and is $\text{Cay}(G, H)$ whenever $H \leq G$ with $|G| = 2n = 2|H|$

but

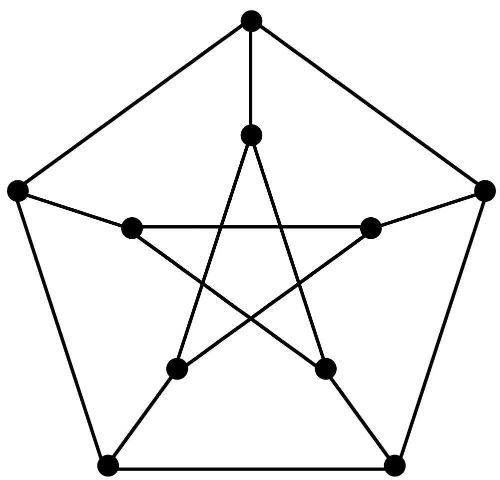
- The Petersen graph, the Gray graph and the Hoffman-Singleton graph are ET but are not Cayley graphs

Bi-Cayley graphs

A bi-Cayley graph is any graph that admits a group H of automorphisms acting with two regular orbits of length $|H|$.

Example 1: $K_{n,n}$, with its two parts being orbits of $H = C_n$

Example 2: the Petersen graph, with two orbits of $H = C_5$ (the vertices of the inner and outer ring, respectively):

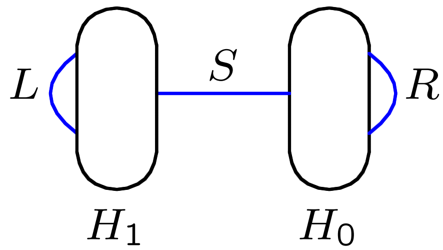


General form of bi-Cayley graphs

Let Γ be a bi-Cayley graph for the group H, which has the two orbits H_0 and H_1 on vertices. We can label the vertices of Γ as h_0 and h_1, where h runs through elements of H.

Then there are subsets R, L and S of H such that the edges of Γ are of three possible forms:

- $\{h_0, (xh)_0\}$ for $x \in R$ and all $h \in H$, joining H_0 to H_0
- $\{h_1, (yh)_1\}$ for $y \in L$ and all $h \in H$, joining H_1 to H_1
- $\{h_0, (zh)_1\}$ for $z \in S$ and all $h \in H$, joining H_0 to H_1.



Construction of bi-Cayley graphs

Let H be any group, and let R, L and S be subsets of H with $|R| = |L|$ such that $1_H \notin R = R^{-1}$ and $1_H \notin L = L^{-1}$.

Define a graph Γ with vertex set being the union $H_0 \cup H_1$ of two copies of H, and with edges of the form $\{h_0, (xh)_0\}$, $\{h_1, (yh)_1\}$ and $\{h_0, (zh)_1\}$ with $x \in R$, $y \in L$ and $z \in S$.

Then this is a bi-Cayley graph, with H acting as a group of automorphisms by right multiplication, and H_0 and H_1 as its two regular orbits on vertices. Also WLOG $1_H \in S$.

We denote this graph by BiCay(H, R, L, S).

Main focus of this work

This talk is a summary of the main points of some recent work, written up in a 27-page paper. This project focussed on bi-Cayley graphs that are edge-transitive, and especially on normal bi-Cayley graphs — namely those where the group induced by H on the vertices of $V(\Gamma)$ is a normal subgroup of the full automorphism group of Γ.

Main theorem

We showed that a finite connected normal edge-transitive bi-Cayley graph can be either arc-transitive, half-arc-transitive or semisymmetric, and moreover, that infinitely many examples of such graphs exist in each case.

[And we found out some other things along the way.]

Two-arc-transitive bi-normal Cayley graphs

Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley graph for a group G, with automorphism group A. Then Γ is a bi-normal Cayley graph if the core $H = \bigcap_{\alpha \in A} G^{\alpha}$ of G in A has index 2 in G. Note that this implies that Γ is a normal bi-Cayley graph over H.

In PAMS 133 (2005), Cai-Heng Li asked for a detailed description of bi-normal Cayley graphs that are 2-arc-transitive, and also whether there are 3-arc-transitive examples.

Our first theorem provides answers to these two questions (on the next slide). For this, we note that every arc-transitive bi-normal Cayley graph is bipartite.

Two-arc-transitive bi-normal Cayley graphs (cont.)

Answers to questions by Li (2005):

Theorem [CFZZ]: Let Γ be a connected bi-Cayley graph $\mathrm{BiCay}(H, \emptyset, \emptyset, S)$. Then $N_{\mathrm{Aut}(\Gamma)}(H)$ acts transitively on the 2-arcs of Γ if and only if the following conditions hold:

(a) $\exists$ an automorphism α of H such that $S^\alpha = S^{-1}$;

(b) the stabiliser of $S \backslash \{1\}$ in $\mathrm{Aut}(H)$ is transitive on $S \backslash \{1\}$;

(c) there exists an automorphism β of the group H such that $S^\beta = s^{-1}S$ for some $s \in S \setminus \{1\}$.

Also $N_{\mathrm{Aut}(\Gamma)}(H)$ is never transitive on the 3-arcs of Γ.

Special cases

For the rest of this project we considered examples and properties of bi-Cayley graphs over abelian groups, dihedral groups and metacyclic groups.

Such graphs may be called bi-abelian graphs, bi-dihedrants, and bi-metacirculants respectively.

In the bi-abelian case, for example, it's easy to prove this

Theorem: Every connected bi-Cayley graph over an abelian group is vertex-transitive.

A family of bi-abelian graphs

Let $H = \langle x \rangle \times \langle y \rangle \cong C_{nm} \times C_m$, where n and m are any two positive integers with $nm^2 \geq 3$, and take $S = \{1, x, x^\lambda y\}$ where $\lambda = 0$ if $n = 1$, or if $n > 1$ take $\lambda \in \mathbb{Z}_n^*$ such that $\lambda^2 - \lambda + 1 \equiv 0 \bmod n$.

Then the 3-valent graph $\Gamma_{m,n,\lambda} = \mathrm{BiCay}(H, \emptyset, \emptyset, \{1, x, x^\lambda y\})$ is always arc-transitive and is sometimes 2-arc-transitive.

Moreover, every connected 3-valent normal edge-transitive bi-Cayley graph over an abelian group is isomorphic to $\Gamma_{m,n,\lambda}$ for some m, n and λ as given above.

These graphs are also important for another reason ...

Trivalent edge-transitive graphs of small girth

A nice by-product of the study of bi-abelian graphs was a complete classification of all 3-valent edge-transitive graphs of girth at most 6. One key part of this was the following:

Theorem: Let Γ be a connected trivalent edge-transitive graph of girth 6. Then either $\Gamma \cong \Gamma_{m,n,\lambda}$ for some m, n and λ with $nm^2 > 9$, or Γ is isomorphic to the Heawood graph, the Pappus graph, the generalised Petersen graph $P(8,3)$, or the generalised Petersen graph $P(10,3)$. In all these cases, the graph Γ is arc-transitive, and also it follows that every connected trivalent semisymmetric graph has girth ≥ 8.

Semisymmetric bi-dihedrants

On this particular topic, we began by proving that if Γ is a connected semisymmetric bi-Cayley graph over the dihedral group $D_n = \langle\, a, b \mid a^n = b^2 = (ab)^2 = 1 \,\rangle$ for some $n \geq 3$, then the valency of Γ is at least 6.

Then we constructed a family of examples of bi-dihedrants, and showed that infinitely many are semisymmetric.

A subclass of the 6-valent examples gave the answer to two open questions posed by Marušič and Potočnik (in Europ. J. Comb (2001)) on tetracirculants — which are graphs that admit an automorphism acting with four cycles of the same length on vertices.

A special family of bi-dihedrants

Let n and k be integers with $n \geq 5$ and $k \geq 2$, for which there exists an element λ of order $2k$ in $\mathbb{Z}_n^*$ such that

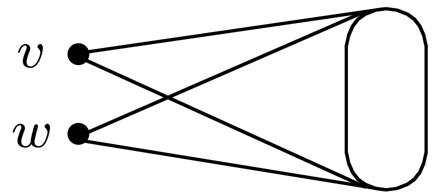
$$1 + \lambda^2 + \lambda^4 + \cdots + \lambda^{2(k-2)} + \lambda^{2(k-1)} \equiv 0 \mod n.$$

Now let $c_i = 1 + \lambda^2 + \lambda^4 + \cdots + \lambda^{2i}$ and $d_i = \lambda c_i$ for all $i \in \mathbb{Z}_k$, and in the dihedral group $D_n = \langle\, a, b \mid a^n = b^2 = (ab)^2 = 1 \,\rangle$, take $S = S(n, \lambda, 2k) = \{a^{c_i} : i \in \mathbb{Z}_k\} \cup \{ba^{d_i} : i \in \mathbb{Z}_k\}$, and define $\Gamma(n, \lambda, 2k) = \mathrm{BiCay}(D_n, \emptyset, \emptyset, S)$.

Then $\Gamma = \Gamma(n, \lambda, 2k)$ is a connected edge-transitive bi-Cayley graph of valency $2k$. Also if k is even and $\lambda^k \equiv -1 \mod n$, then Γ is arc-transitive, but if k is odd and $\lambda^k \equiv -1 \mod n$, then Γ is semisymmetric.

Digression: Worthy and unworthy graphs

A graph is called unworthy [thanks to Steve Wilson] if two of its vertices v and w have exactly the same neighbours:



In this case, there exists an automorphism of the graph that swaps v and w but fixes all others. It follows that the stabiliser of an edge not incident with v or w is non-trivial.

Hence (in particular), every edge-regular graph is 'worthy'.

Answer to questions by Marušič and Potočnik

A tetracirculant is a graph that admits an automorphism θ acting with four cycles of the same length on vertices.

In their 2001 paper, Marušič and Potočnik constructed a family of 'generalised Folkman tetracirculants', and asked:

Does this family include every semisymmetric tetracirculant?
or at least every semisymmetric tetracirculant Γ such that the four orbits of $\langle\theta\rangle$ are blocks of imprimitivity for $\mathrm{Aut}(\Gamma)$?

Generalised Folkman tetracirculants are unworthy, but if $\lambda^3 \not\equiv -1 \bmod n$, then our graph $\Gamma(n, \lambda, 6)$ is a semisymmetric tetracirculant that is edge-regular and hence 'worthy'. Also for $\Gamma(n, \lambda, 6)$ the four orbits of $\langle\theta\rangle$ are blocks of imprimitivity, and so the answer to both questions above is 'No'.

Metacirculants

In doing this work, we were led to study a question posed in JACo 28 (2008) by Marušič and Šparl about metacirculants.

A graph is a weak metacirculant if it admits a metacyclic group of automorphisms acting transitively on vertices.

A graph Γ is an (m, n)-metacirculant if it has order mn, and admits two automorphisms σ and τ such that

- σ has order n, and $\langle\sigma\rangle$ acts semi-regularly on $V(\Gamma)$, while
- τ has order divisible by m, has a cycle of size m on $V(\Gamma)$, normalises $\langle\sigma\rangle$, and cyclically permutes the m orbits of $\langle\sigma\rangle$.

Clearly every (m, n)-metacirculant is a weak metacirculant.

Question: Is the converse also true?

Answer 1:

Li, Song and Wang claimed the converse is false, in a theorem in a paper in JCTA (2013) stating that every non-split metacyclic p-group (for p an odd prime) acts transitively on the vertices of a half-arc-transitive 4-valent graph Γ, such that Γ is a weak metacirculant but not a metacirculant.

Unfortunately they made a mistake in the first paragraph of the proof of their main Theorem, and the theorem is wrong.

In fact:

Theorem [CFZZ]: Let Γ be any connected 4-valent half-arc-transitive graph of order p^n for some odd prime p. Then Γ is weak metacirculant if and only if Γ is a metacirculant.

Answer 2 [Šparl and Antončič]:

Šparl and Antončič found two 4-valent weak metacirculants of order 160 that are not metacirculants in the census of all 4-valent half-arc-transitive graphs up to order 1000 created by Potočnik, Spiga and Verret (2015).

Answer 3 [CPZZ]:

Let $\Gamma = \text{BiCay}(H, R, L, S)$, where $H = \langle a \rangle = C_{28}$, and $R = \{a, a^{-1}\}$, $L = \{a^{13}, a^{-13}\}$ and $S = \{1, a, a^6, a^{19}\}$.

Then Γ has valency 6, and an easy MAGMA computation shows Γ is half-arc-transitive, with $\text{Aut}(\Gamma) \cong (C_7 \times Q_8) \rtimes C_3$.

In particular, Γ is weak metacirculant (of order 56), but it is not difficult to show that it is not a metacirculant.

Answer 3b [CPZZ]:

The last example is the first member of an infinite family of examples of 'non-metacirculant' weak metacirculants.

Let p be any prime congruent to 1 mod 6, and let $H = \langle x \rangle$ be the cyclic group C_{4p} of order $4p$, with generator x. Next let r be any square root of -3 mod p, and use the CRT to find an integer t such that $t \equiv r$ mod p and $t \equiv 3$ mod 4 — e.g. $(r, t) = (6, 19)$ when $p = 13$.

If $R = \{x, x^{-1}\}$, $L = \{x^{2p-1}, x^{2p+1}\}$ and $S = \{1, x, x^t, x^{2p+t+1}\}$, then BiCay$(H, R, L, S)$ is a 6-valent half-arc-transitive weak metacirculant with automorphism group $(C_p \times Q_8) \rtimes C_3$, but is not an (m, n)-metacirculant for any m and n.

Bi-metacirculants

A bi-metacirculant is a bi-Cayley graph over a metacyclic group, and a bi-p-metacirculant is a bi-Cayley graph over a metacyclic p-group.

— e.g. the smallest graph $B(2,6,9)$ in a family of half-arc-transitive graphs constructed by Bouwer (1970) is a 4-valent bi-Cayley graph over a metacylic group of order 27.

Theorem [CFZZ]: If Γ is a 4-valent vertex- and edge-transitive bi-Cayley graph over a non-abelian metacyclic p-group H, then $R(H)$ is normal in Aut(Γ).

Theorem [CFZZ]: There exist 4-valent half-arc-transitive bi-p-metacirculants of order $2p^3$, for every odd prime p.

Construction

For any odd prime p, let H be the metacyclic group of order p^3 with presentation $\langle\, a, b \mid a^{p^2} = b^p = 1,\ b^{-1}ab = a^{1+p} \,\rangle$, and then let $\Gamma_p = \mathrm{BiCay}(H, \emptyset, \emptyset, S)$ where $S = \{1, a^2, a^p b^2, a^{2-p} b^2\}$.

Then Γ_p is a 4-valent half-arc-transitive bi-p-metacirculant (of order $2p^3$) over $C_{p^2} \rtimes_{1+p} C_p$, and is also an edge-regular Cayley graph for every such p.

Note: There is also another family of 4-valent half-arc-transitive bi-p-metacirculants based on the same group when $p \equiv 1$ modulo 4, but these graphs are not Cayley graphs.

Summary

Arc-transitive case: There exists an arc-transitive 3-valent bi-Cayley graph over $C_{nm} \times C_m$ for any m and n with $nm^2 \geq 3$

Semi-symmetric case: There exist semisymmetric bi-dihedrants of valency $2k$ for every odd integer $k \geq 3$

Half-arc-transitive case: There exist 4-valent half-arc-transitive bi-p-metacirculants of order $2p^3$ for every odd prime p

... and answers to some open questions along the way.

THANK YOU

Title: Bi-Cayley graphs

Speaker: Marston Conder, University of Auckland

Abstract:

Cayley graphs form an important class of vertex-transitive graphs, which have been the object of study for many decades. These graphs admit a group of automorphisms that acts regularly (i.e. sharply-transitively) on vertices. On the other hand, there are many important vertex-, edge- or arc-transitive graphs that are not Cayley graphs, such as the Petersen graph, the Gray graph, and the Hoffman-Singleton graph.

In this talk, I will describe some recent developments in the theory of *bi-Cayley graphs*, which are graphs that admit a group H of automorphisms acting semi-regularly on the vertices, with two orbits (of the same length). These include the Petersen graph and the Gray graph, and many more besides.

We focus mainly on the case where the group H is normal in the full automorphism group of the graph, and have produced infinite families of examples in each of three subclasses of bi-Cayley graphs, namely those that are arc-transitive, half-arc-transitive or semisymmetric, respectively.

In doing this, we found the answer to a number of open questions about these and related classes of graphs, posed by Li (in *Proc. American Math. Soc.* 133 (2005)), Marušič and Potočnik (in *European J. Combinatorics* 22 (2001)) and Marušič and Šparl (in *J. Algebraic Combinatorics* 28 (2008)). Also we found and corrected an error in a recent paper by Li, Song and Wang (in *J. Combinatorial Theory, Series A* 120 (2013)).