Base Size and Generic Stabilizers

Robert M. Guralnick

University of Southern California

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Setup

G is a group acting on a set X of cardinality > 1
usually assume G is faithful (or reduce to that case)

Definition 1

A base for G on X is a subset B of X such that $g \in G$ trivial on B implies g is trivial on X.

The *base size* is the minimal cardinality of a base.
If b is small, then so is G.

Note that if $|X| = n$, then $|G| \leq b^n$

If X is a variety, then $\dim G \leq b \dim X$.

Moreover, computationally having a small base is very efficient.

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It is a classical problem to determine base size.
Examples:

1. $b(S_n) = n - 1$, $b(A_n) = n - 2$
2. $b(GL(n, k)) = n$
3. $b(PGL(n, k), Gr_d) = n/d + 2$

In the last case, we assume that $1 < d < n/2$ and d divides n.

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Based on results from several papers, we have:

Theorem 2

If G is a finite simple group acting primitively on X, then either the action is "standard" or $b \leq 7$.

There is one example of base size 7 but infinitely many of base size 6 (coming from algebraic groups).

A relatively recent approach – instead of producing a base, show most subsets of size b are a base. Used by

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Related problem: We say G has a regular orbit on X if $G_x = 1$ for some $x \in X$.

Note that G has base size b on X if and only if G has a regular orbit on X^b.

Regular orbits come up in many situations – in particular in the $k(GV)$ problem of Brauer.

Ongoing program to determine the base size for G almost simple acting primitively on X. Done in many cases by various authors.
From now on, we will assume G is a simple algebraic group and X is an irreducible variety. There is a close relation between the base size for the algebraic group and the base size for the finite simple groups of Lie type.

We say a generic stabilizer has a given property if this holds for a nonempty open subset of the variety.

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We consider two main cases:

1. $X = G/H$ where H is a maximal closed subgroup of G.
2. X is a (rational) irreducible finite dimensional G-module.

New invariants in addition to b; b^0 is the smallest size where some (and so generic) stabilizer of b^0 points is finite
b^1 is the smallest size where the generic stabilizer is trivial

Clearly $b^0 \leq b \leq b^1$.
Burness-Guralnick-Saxl have determined in almost all cases in case (1) b^0, b, b^1. Often they are all 2. We use the classification of maximal closed subgroups.

Example: Assume characteristic not 2, τ an involution inverting a maximal torus T. $H = C_G(\tau)$ and $X = G/H$. Then generically the stabilizer of two points is conjugate to $T[2]$ (2-torsion in T). However, there always is a regular orbit on X^2.

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In char 0, studied by A. M. Popov, Vinberg, V. Popov, etc.

Richardson: If G is reductive and X is a smooth affine variety in characteristic 0, then generic stabilizers exist (i.e. there is an open sub variety where all point stabilizers are conjugate). Fails in positive characteristic.

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Another example (question of Reichstein) H is the normalizer of a torus. $X = G/H$. Then $b = b^1 = b^0 = 2$ unless $G \cong PGL_2$ in which case generically a stabilizer in X^2 has order 2

Main tool: try to show that for any $g \in G$ of prime order (or unipotent in char 0), we have $\dim V^g + \dim g^G < \dim V$.

For some applications, you want to know that G_x is generically trivial as a group scheme. This can be checked by considering the Lie algebra of G.

BGLT: if $\dim V > \dim G$, then G_x is generically finite.

BGLT and Garibaldi-G show that usually if $dim V > \dim G$, then the generic stabilizer is trivial as a group scheme.

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The cases where G_x is generically finite but nontrivial are quite interesting. This has been used by Bhargava (in char 0).

Some examples:

1. $SO_n, n \geq 7$ on $L(2\lambda_1)$, characteristic not 2
2. $HSpin_{16}$ on the half spin module of dimension 128
3. SL_9 on $\wedge^3$
4. SL_8 on $\wedge^4$
5. a few more small dimensional examples

Many examples come from the Vinberg setup.
For example, if $G = SO_n$, we consider SL_n acting on the symmetric square. A generic stabilizer is SO_n and so a generic stabilizer in SO_n on $L(2\lambda_1)$ is the intersection of SO_n with a generic conjugate in SL_n – note that SO_n is the centralizer of an involution inverting a maximal torus. So a generic stabilizer is the 2-torsion in a maximal torus.

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In characteristic 0, if the generic stabilizer is finite, then there is a regular orbit (usually unique).

In positive characteristic, this is usually true. Two counterexamples:

1. $G = SL_4$ in characteristic 3 and $\dim V = 20$;
2. $G = SL_n$ and $V = L \otimes L^{(q)}$ or $L^* \otimes L^{(q)}$ where L is the natural module and $L^{(q)}$ is the Frobenius twist.

A result noted in Claborn, Kirwan, Mumford: $\mathbb{C}[V]^G$ is free if and only if the generic stabilizer is not trivial (proof is by comparing lists).

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Theorem 3

If G is a simple group of adjoint type of rank at least 2, then the essential dimension of G is at most $\dim G - 2\mathrm{rank}G - 1$.

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It turns out for most simple algebraic groups over an algebraically closed field, the essential dimension is bounded above by dim G. The exception comes out of a beautiful result of Brosnan-Reichstein-Vistoli who give exponential lower bounds for Spin and Half Spin groups (in characteristic not 2) and essentially show in characteristic 0, the lower bounds are the right answer (for $n \geq 15$).

In positive odd characteristic, this is also true. It follows from GG that for $15 < n \neq 16$, the half spin or spin groups act generically freely on the half spin or spin modules. This uses the inequality $\mathrm{ed}(G) \leq \dim X - \dim G$ where G acts generically freely on the variety X and the action is verbal (any linear module is verbal).

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Another application of generic stabilizers: stabilizers of homogeneous polynomials.

Theorem 4

Let H be a simple algebraic group with V an irreducible module. Suppose that $H < G \leq SL(V)$. Then with a small number of exceptions, $k[V]^G$ has smaller transcendency degree than $k[V]^H$.

This implies that for almost all $f \in k[V]^H$, H is the connected component of the stabilizer in $GL(V)$ of f.

A very special case is a 125 year old question of Cartan (new even over $\mathbb{C}$).

Let f be the degree 8 invariant of $E_8(k)$ acting on its Lie algebra. Then the stabilizer in $GL(V)$ of f is just $\mu_8 \times E_8(k)$.

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