An algebraic topologist ponders the ring of quasisymmetric functions and variants

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The theme

Algebras, coalgebras, and Hopf algebras of combinatorial interest occur as the cohomology and homology of spaces familiar to algebraic topologists.

- Decompositions of spaces $\Rightarrow$ decompositions of algebras
- Filtrations of spaces $\Rightarrow$ filtrations of algebras
- Homotopy equivalences of spaces $\Rightarrow$ isomorphisms of algebras
- 'Global' perspective of algebraic topology gives insight into algebraic structure.

I'll illustrate this with a 'Freeness Theorem' and an 'Integrality Theorem'.

Remark I am using ideas and the vibe of [Andy Baker and Birgit Richter, Quasisymmetric functions from a topological point of view, Math. Scan. **103** (2008)].

Three graded Hopf algebras

Symmetric functions: $Symm = \mathbb{Z}[e_1, e_2, e_3, \dots]$, with $|e_k| = 2k$.

This is a Hopf algebra: $\Delta(e_k) = \sum_{i+j=k} e_i \otimes e_j$. ($e_0 = 1$.)

It is bicommutative and self dual.

Non commuting symmetric functions: $NSymm = T\langle e_1, e_2, e_3, \dots \rangle$.
It is cocommutative.

Quasisymmetric functions: $QSymm = (NSymm)^*$, the dual Hopf algebra.
It is commutative.

A longtime conjecture finally proved around 2000:

Theorem $QSymm$ is free commutative (i.e., polynomial) as an algebra.

Three graded Hopf algebras, topologically realized

- $Symm = H_*(BU) = H^*(BU)$.

BU = classifying space for (virtual) complex vector bundles. The coproduct on $Symm$ is induced by the direct sum of vector bundles.

- $NSymm = H_*(\Omega\Sigma\mathbb{CP}^\infty)$ and $QSymm = H^*(\Omega\Sigma\mathbb{CP}^\infty)$.

$\mathbb{CP}^\infty = BU(1)$ is infinite dimensional complex projective space.

ΣX is the suspension of a based space X. $\widetilde{H}_{n+1}(\Sigma X) = \widetilde{H}_n(X)$.

ΩY is the space of based loops in a based space Y.

Σ and Ω are adjoint, so $\Omega\Sigma X$ = the free loopspace generated by X.

An 'evident' map $\Omega\Sigma\mathbb{CP}^\infty \to BU$ induces the evident maps

$$NSymm \twoheadrightarrow Symm \text{ and } Symm \hookrightarrow QSymm.$$

Some categories

Work with nonneg. graded abelian groups, f.g. and free in each degree.

- Alg = commutative algebras A with $A_0 = \mathbb{Z}$. So A is augmented.

$I(A)$ = the augmentation ideal = positive part of A.

- $Coalg$ = coaugmented cocommutative coalgebras C with $C_0 = \mathbb{Z}$.

$J(C)$ = coaugmentation coideal = positive part of C.

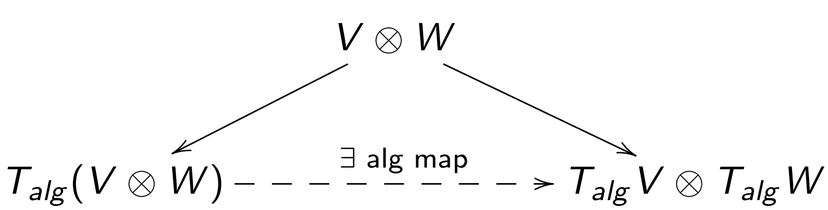
- $Hopf$ = Hopf algebras, with subcategories $Hopf^{com}$, $Hopf^{cocom}$.

Variants $Alg_{\mathbb{F}}$, etc.: everything over a field $\mathbb{F}$.

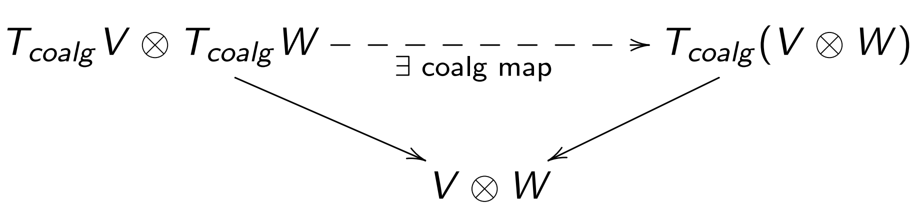
Tensor functors

With V a positively graded, $TV = \bigoplus_{m=0}^{\infty} V^{\otimes m}$.

Let $T_{alg}V = TV$, viewed as the free algebra generated by V.



Let $T_{coalg}V = TV$, viewed as the cofree coalgebra cogenerated by V.



Tensor functors (cont.)

- $C \in Coalg \Rightarrow T_{alg}(J(C)) \in Hopf^{cocom}$ with comultiplication:

$$T_{alg}(JC) \xrightarrow{T(\Delta_C)} T_{alg}(JC \otimes JC) \to T_{alg}(JC) \otimes T_{alg}(JC).$$

- $A \in Alg \Rightarrow T_{coalg}(I(A)) \in Hopf^{com}$ with multiplication:

$$T_{coalg}(IA) \otimes T_{coalg}(IA) \to T_{coalg}(IA \otimes IA) \xrightarrow{T(m_A)} T_{coalg}(IA).$$

Example Let $E = \langle e_0, e_1, e_2, \ldots \rangle$, $|e_k| = 2k$, $\Delta(e_k) = \sum_{i+j=k} e_i \otimes e_j$.

Then $T_{alg}(J(E)) = NSymm$.

Dual Example $E^* = \mathbb{Z}[c]$, $|c| = 2$. $T_{coalg}(I(E^*)) = QSymm$.

QSymm is polynomial

Freeness Theorem If A is free commutative, then $T_{coalg}(I(A))$ is free commutative as an algebra.

Specializing to $A = \mathbb{Z}[c]$, $|c| = 2$, we deduce

Corollary *QSymm* is free commutative as an algebra.

Remark First correct proof by Hazewinkel in Adv.Math. in 2001. (Or maybe Crossley in Bull.L.M.S. in 2000?)

I'll outline an elementary algebraic proof of the theorem inspired by topology (after Baker-Richter) ...

Proof of the Freeness Theorem

Lemma $B \in Alg$ is free commutative $\Leftrightarrow B_{\mathbb{F}} = B \otimes \mathbb{F}$ is free commutative in $Alg_{\mathbb{F}}$ for all prime fields $\mathbb{F}$.

Lemma [Milnor-Moore] As an algebra, an evenly graded $H \in Hopf_{\mathbb{F}}^{com}$ is always free if $\mathbb{F} = \mathbb{Q}$, and is the tensor product of algebras of the form $\mathbb{F}_p[x]$ and $\mathbb{F}_p[x]/(x^{p^r})$, if $\mathbb{F} = \mathbb{F}_p$.

Definition For $A \in Alg_{\mathbb{F}_p}$, let $\Phi : A \to A$ be the pth power map.

Remark Φ is linear!

Corollary $H \in Hopf^{com}$ is free as a commutative algebra $\Leftrightarrow$ $\Phi : H_{\mathbb{F}_p} \to H_{\mathbb{F}_p}$ is monic for all primes p.

A hint from topology

So the Freeness Theorem will follow from ...

Proposition Given $A \in Alg_{\mathbb{F}_p}$, if $\Phi : A \to A$ is monic, so is $\Phi : T_{coalg}(I(A)) \to T_{coalg}(I(A))$.

To prove this, we get a hint from topology ...

For X a based space, $H_*(\Omega\Sigma X) = T_{alg}(\widetilde{H}_*(X))$, $H^*(\Omega\Sigma X) = T_{coalg}(\widetilde{H}^*(X))$, and $\Omega\Sigma X$ has a filtration with 'subquotients' $X^{\wedge n}$ = the n–fold smash product of X with itself.

All known, and mainly due, to Ioan James in the early 1950's.

Lets model the filtration algebraically ...

Proof of the Freeness Theorem (cont.)

Work over $\mathbb{F}_p$. Let $T^{\leq n}V = \bigoplus_{m=0}^{n} V^{\otimes m}$.

Easy to check: $T^{\leq n}_{alg}(J(C))$ is a subcoalgebra of $T_{alg}(J(C))$.

Dually: $T^{\leq n}_{coalg}(I(A))$ is a quotient algebra of $T_{coalg}(I(A))$.

$$0 \to I(A)^{\otimes n} \to T^{\leq n}_{coalg}(I(A)) \to T^{\leq (n-1)}_{coalg}(I(A)) \to 0$$

is a short exact sequence of graded $\mathbb{F}_p$–vector spaces with linear endomorphism Φ.

$$\begin{aligned} \Phi : A \to A \text{ is monic} &\Rightarrow \Phi : I(A)^{\otimes n} \to I(A)^{\otimes n} \text{ is monic } \forall n \\ &\Rightarrow \Phi : T^{\leq n}_{coalg}(I(A)) \to T^{\leq n}_{coalg}(I(A)) \text{ is monic } \forall n \\ &\Rightarrow \Phi : T_{coalg}(I(A)) \to T_{coalg}(I(A)) \text{ is monic.} \end{aligned}$$

More fun

The primitive coalgebra: Let $P = \langle p_0, p_1, p_2, \ldots \rangle$ with

$$\Delta(p_k) = p_k \otimes 1 + 1 \otimes p_k.$$

$T_{alg}(J(P)) = T\langle p_1, p_2, \ldots \rangle$. Let *Shuf* be its dual: the shuffle algebra.

Hazewinkel (and others) note that the formula

$$\sum_{k=0}^{\infty} e_k t^k = \exp(\sum_{k=1}^{\infty} p_k t^k)$$

defines an isomorphism of Hopf algebras over $\mathbb{Q}$

$$NSymm_{\mathbb{Q}} = T_{alg}(J(E))_{\mathbb{Q}} \simeq T_{alg}(J(P))_{\mathbb{Q}}.$$

Not true over $\mathbb{Z}$. But how close?

An integrality theorem

One more coalgebra: Let $D = \langle d_0, d_1, d_2, \dots \rangle$ where $d_k = k! e_k$.
Then $D \subset E$ as coalgebras, and so $T_{alg}(J(D)) \subset T_{alg}(J(E)) = NSymm$ as Hopf algebras. Clearly this is a rational isomorphism.

Integrality Theorem The formula $\sum_{k=0}^{\infty} e_k t^k = \prod_{k=1}^{\infty} \exp(\frac{p_k}{k!} t^k)$ defines an isomorphism of Hopf algebras over $\mathbb{Q}$

$$T_{alg}(J(E))_{\mathbb{Q}} \simeq T_{alg}(J(P))_{\mathbb{Q}},$$

which restricts to an isomorphism of Hopf algebras over $\mathbb{Z}$

$$T_{alg}(J(D)) \simeq T_{alg}(J(P)).$$

Corollary *QSymm* embeds in *Shuf* as a sub-Hopf algebra.

Where did this come from?

$E = H_*(\mathbb{CP}^\infty)$, $D = H_*(\Omega S^3)$, $P = H_*(\bigvee_{k=1}^{\infty} S^{2k})$.

A friendly map $\Omega S^3 \to \mathbb{CP}^\infty$ induces $D \subset E$.

Theorem (James) There is a natural homotopy equivalence

$$\Sigma\Omega\Sigma X \simeq \Sigma(\bigvee_{k=1}^{\infty} X^{\wedge k}).$$

In particular, $\Sigma\Omega S^3 \simeq \Sigma(\bigvee_{k=1}^{\infty} S^{2k})$. Now apply Ω to see that there is an equivalence $\Omega\Sigma\Omega S^3 \simeq \Omega\Sigma(\bigvee_{k=1}^{\infty} S^{2k})$, as loopspaces.

This induces a Hopf algebra isomorphism $T_{alg}(J(D)) \simeq T_{alg}(J(P))$.

The explicit formula comes from analyzing $\Omega S^3 \to \Omega\Sigma(\bigvee_{k=1}^{\infty} S^{2k})$, a sum of James-Hopf maps, in homology.

Contrasting examples

Under Hazewinkel's rational isomorphism, one has:

$$\begin{aligned}
d_1 &= p_1 \\
d_2 &= p_1^2 + 2p_2 \\
d_3 &= p_1^3 + 3p_1p_2 + 3p_2p_1 + 6p_3 \\
d_4 &= p_1^4 + 4p_1^2p_2 + 4p_1p_2p_1 + 4p_2p_1^2 + 12p_2^2 + 12p_1p_3 + 12p_3p_1 + 24p_4
\end{aligned}$$

Under my integral isomorphism, one has:

$$\begin{aligned}
d_1 &= p_1 \\
d_2 &= p_1^2 + p_2 \\
d_3 &= p_1^3 + 3p_1p_2 + p_3 \\
d_4 &= p_1^4 + 6p_1^2p_2 + 3p_2^2 + 4p_1p_3 + p_4
\end{aligned}$$

In both cases, $\Delta(d_k) = \sum_{i+j=k} \frac{k!}{i!j!} d_i \otimes d_j$.

Thanks for listening!