The Hurwitz action in real reflection groups

(w/ Vic Reiner)

G a group

Hurwitz move

$$(t_1, \ldots, t_i, t_{i+1}, \ldots, t_m) \in G^m$$

Hurwitz action

$$\downarrow \sigma_i$$

Braid group action

$$(t_1, \ldots, t_{i+1}, t_{i+1}^{-1} t_i t_{i+1}, \ldots, t_m)$$

<u>Obs</u>: preserves product $t_1 \cdots t_m =: c$, so acts on <u>factorizations</u>

Question: connectedness? (Equiv: orbit structure.)

(Background:

- Hurwitz, 1890s, $G = S_n$, t_i transpositions,
  something to do w/ covering spaces of the Riemann sphere
- Bessis, 2000s, $G =$ finite real refl'n group, t_i reflections,
  dual Coxeter systems, eventually something to do w/ $K(\pi, 1)$ Eilenberg-MacLane for complement of $\mathbb{C}$ hyperplane arrangement of $\mathbb{C}$ refl'n gps.)

Thm (Bessis; "dual Matsumoto-Tits lemma"): W a finite real refl'n gp of rank n, $c \in W$ a Coxeter element. The Hurwitz action on minimal (length n) factorizations of c into reflections is transitive.

Finite real refl'n gp: matrix gp (finite, over $\mathbb{R}$) generated by its subset of orthogonal reflections thru a hyperplane.

Generators & relations:

$$\langle s_1, \ldots, s_n \mid s_i^2 = 1,\ (s_i s_j)^{} = 1 \rangle$$

"simple refl'ns"

Coxeter el't: $c = s_1 s_2 \cdots s_n$ (& conjugates)

E.g.: • S_n: perm matrices, s_i = adjacent transpositions, $c = (1\,2\,3\,4 \cdots n)$

• type B: hyperoctahedral gp / signed perm. matrices, $c = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & & n & -1 \end{pmatrix}$ (two-line)

$$= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & & 0 & 0 \\ 0 & 1 & 0 & & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ & & \vdots & & \vdots & \vdots \\ & & & & 1 & 0 \end{pmatrix}$$

(as a matrix)

E.g. S_3

$$c = (123) = \overset{s_1}{(12)}\overset{s_2}{(23)}$$

$$\begin{tikzcd}
\overset{s_1}{(12)}\overset{s_2}{(23)} \arrow[d, "\sigma_1"] \\
= (23)(13) \arrow[d, "\sigma_1"] \\
= (13)(12) \arrow[uu, bend right=60, "\sigma_1"']
\end{tikzcd}$$

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Thm (L–Reiner) W, c, ditto. $m \in \mathbb{N}$. The Hurwitz action on factorizations of c as a product of m reflections is as transitive as possible.

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(two-line)

$$= \begin{pmatrix} & & [illegible] & & \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & & \vdots & & \vdots & \vdots \\ & & & & 1 & 0 \end{pmatrix}$$

(as a matrix)

E.g. S_3

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\end{tikzcd}$$

E.g. Type B_2

$$W = \left\{ \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}, \begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix} \right\} \quad (8 \text{ elmts})$$

Reflections

$$\overset{s_2}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \qquad \overset{s_1}{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$C = \overset{s_1 s_2}{\begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix}} = s_1 \cdot s_1 \cdot s_1 \cdot s_2$$

$$= s_2 \cdot s_2 \cdot s_1 \cdot s_2$$

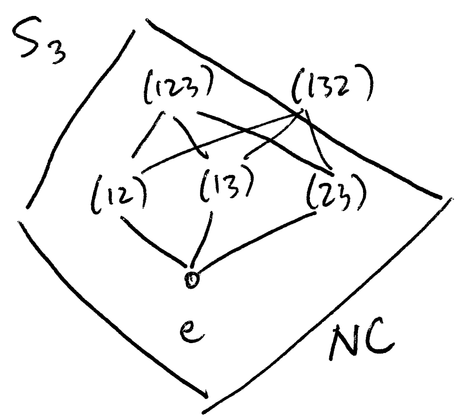
An idea from Bessis' pf (& ~~[illegible]~~ related work):

view $(t_1, \ldots, t_m)$ as a <u>walk</u>

$$e \to t_1 \to t_1t_2 \to \cdots \to t_1t_2\cdots t_m$$

on W.

Elements we meet on <u>shortest</u> paths to c are "W-noncrossing partitions"



But if $m > n$, we don't need to stay in $NC(W)$!

So we <u>cannot</u> use existing tools for NC.

<u>Structure of pf:</u>

Main Thm

$\uparrow$

Lemma 1 : factorizations of <u>arbitrary</u> $w \in W$

$\uparrow$

Lemma 2 : root circuits are acutely disconnected

<u>Lemma 1</u> (L-Reiner) W a finite real reflection group, w an arbitrary element of W, $t = (t_1, \ldots, t_m)$ a factorization of w into reflections. The Hurwitz orbit of t contains a factorization

$$(t_1', t_2', t_3', t_4', \ldots, t_{2k-1}', t_{2k}', t_{2k+1}', \ldots, t_m') \text{ s.t.}$$

$$t_1' = t_2', \; t_3' = t_4', \ldots, t_{2k-1}' = t_{2k}' \;\&\; (t_{2k+1}', \ldots, t_m') \text{ is}$$

a <u>shortest</u> factorization of w.

Use this to put long factorizations into semi-canonical form; then use Bessis' results about shortest factorizations of Cox. elts to finish.

Root systems

W a finite real refl'n gp.
Take a pair $\pm\alpha$ of vectors $\perp$ to the reflecting hyperplane of each reflection in W.
This is a root system
$\Phi = \Phi(W)$.

(Also axioms ...)

Lemma 2 (L-Reiner)

Suppose $\{\alpha_1, \ldots, \alpha_k\}$ is a circuit (i.e., minimal dependent set) in a root system. Replace α_i by $-\alpha_i$ as needed s.t. dependence

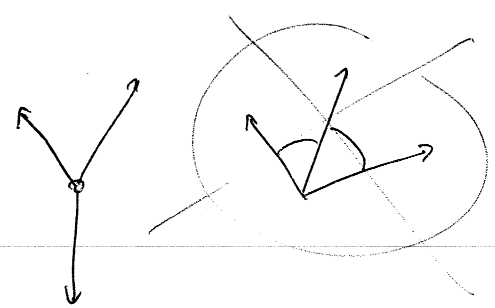
$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_k\alpha_k = 0$$

has $c_i > 0$. Define <u>acuteness graph</u> G by
$$V(G) = \{1, \ldots, k\},$$
$$E(G) = \{ \{i,j\} : \alpha_i \cdot \alpha_j > 0 \}.$$

Then G is <u>disconnected</u>.

Rmk (Fiedler): if α_i not restricted to a root system, obtuseness graph must be connected, acuteness graph is arbitrary.

How to prove acuteness graphs are disconnected?

Rk 2: 

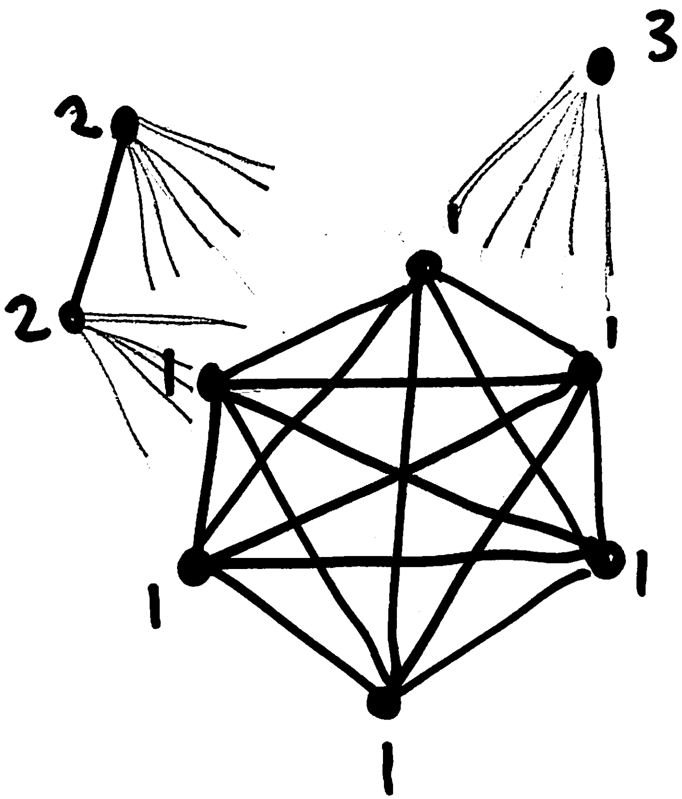
Classical types A, B, C, D: Zaslavsky, signed graphs, circuits are understood, very few edges in acuteness graph.

Exceptional types: weeks of computer computations :(.

Lots of nice data produced; interested?

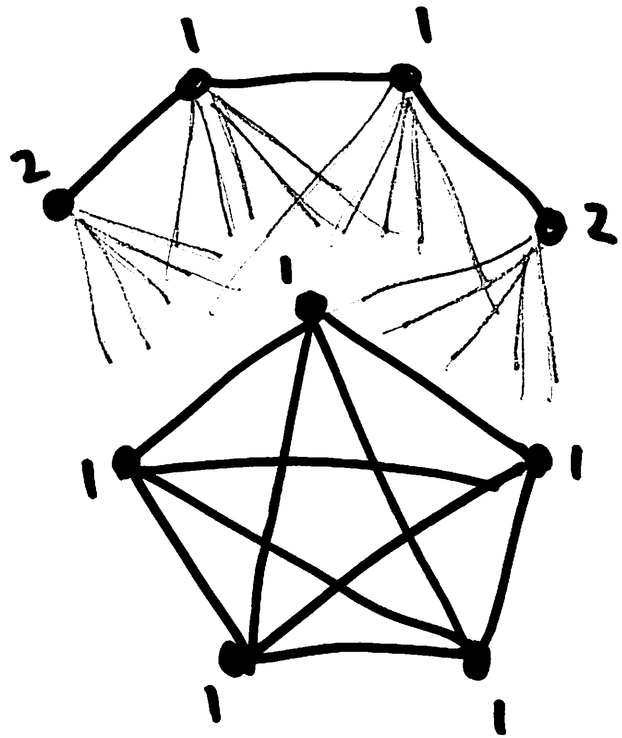
$\mathcal{E}_8$ examples

16 acute edges, 3 components



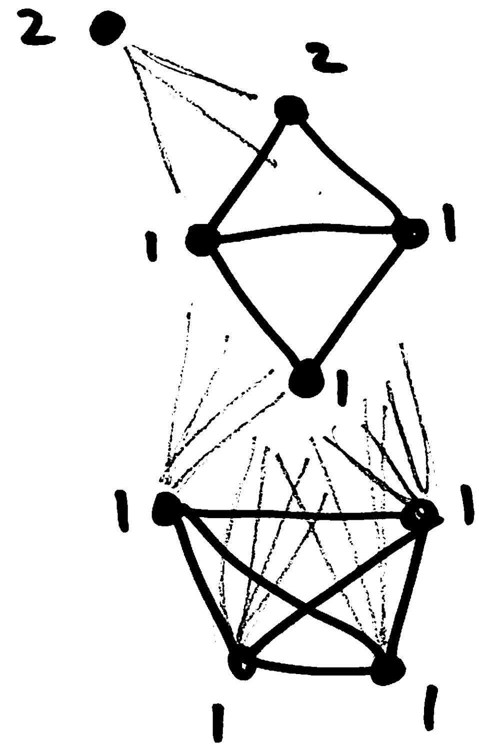
(18 obtuse edges)

13 acute edges, 2 components



(20 obtuse edges)

11 acute edges, 3 components,



(19 obtuse edges)