# Coxeter-biCatalan combinatorics

Nathan Reading

NC State University

Algebraic Combinatorics and Group Actions
Herstmonceux, July 14, 2016



*Joint work with Emily Barnard* (arXiv:1605.03524)

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Ackerman-Barequet-Pinter '06: (Diagonal) rectangulations.

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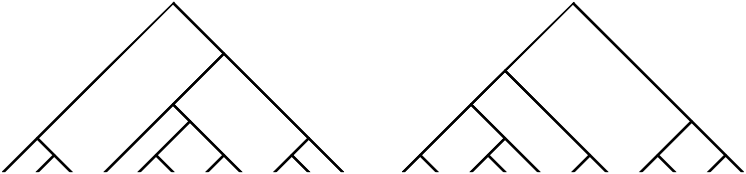

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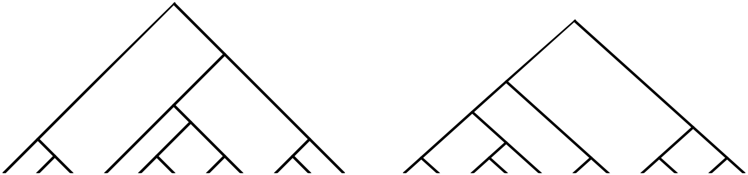

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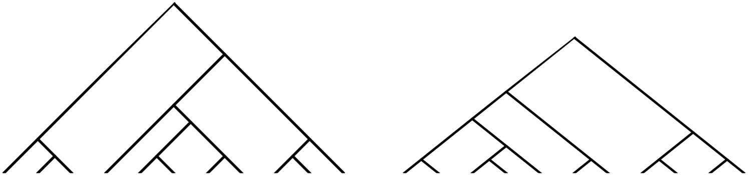

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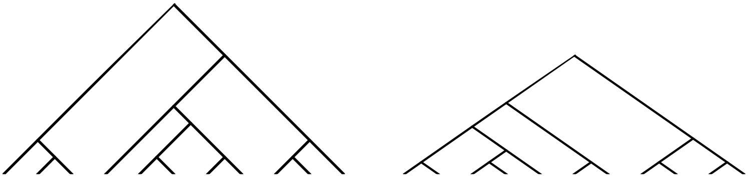

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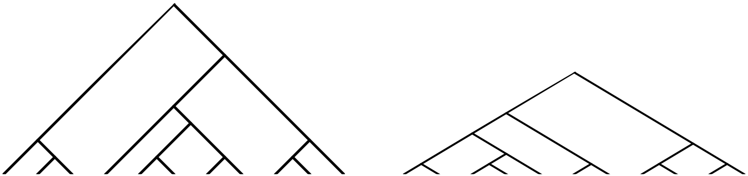

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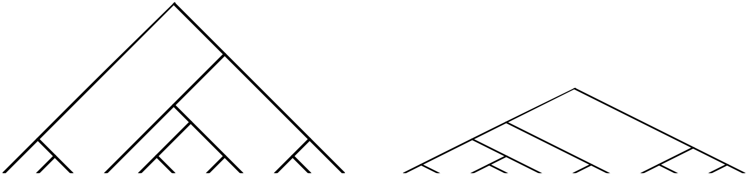

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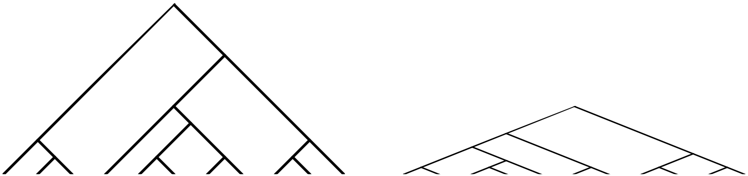

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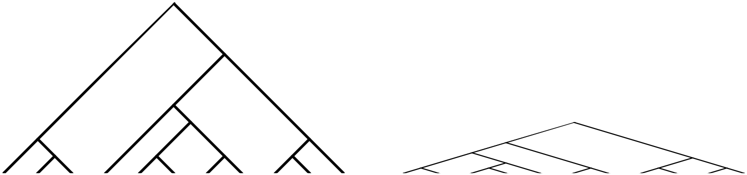

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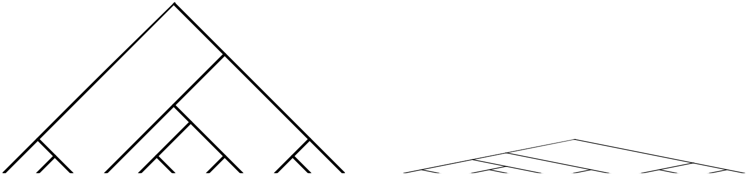

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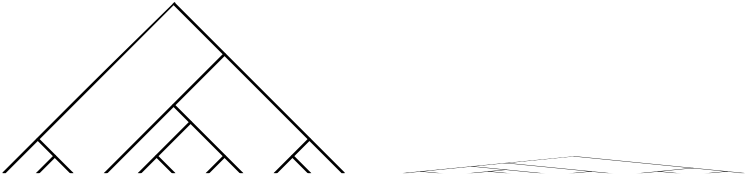

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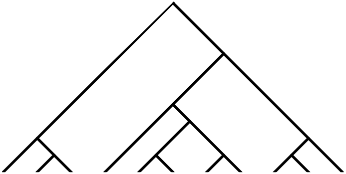

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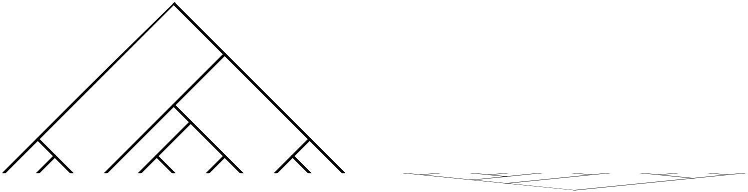

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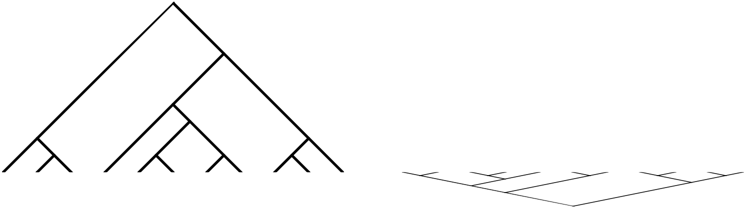

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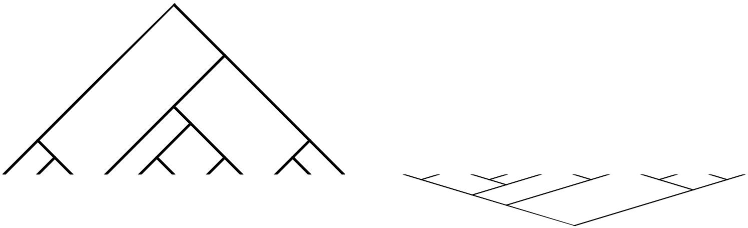

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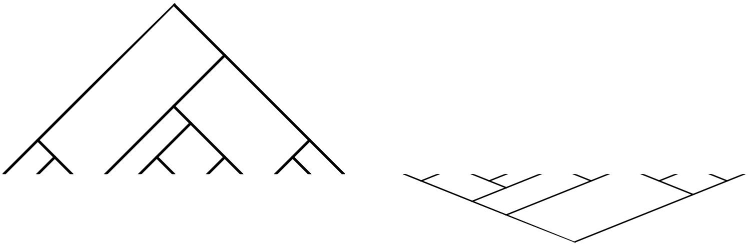

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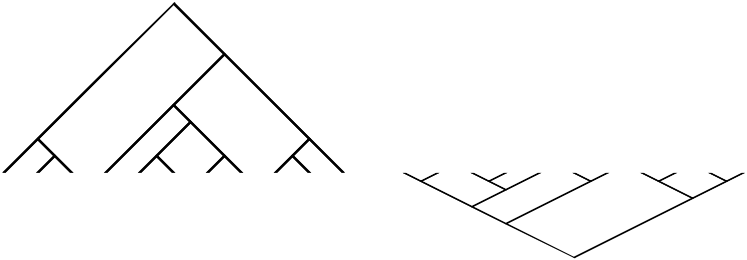

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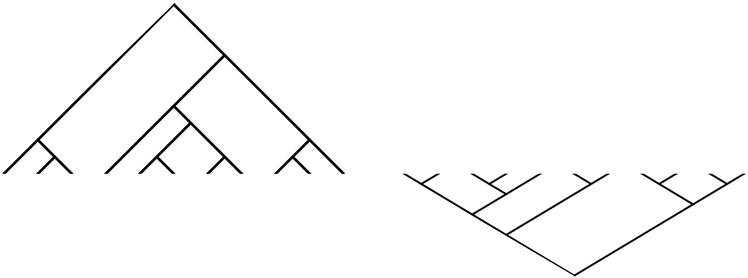

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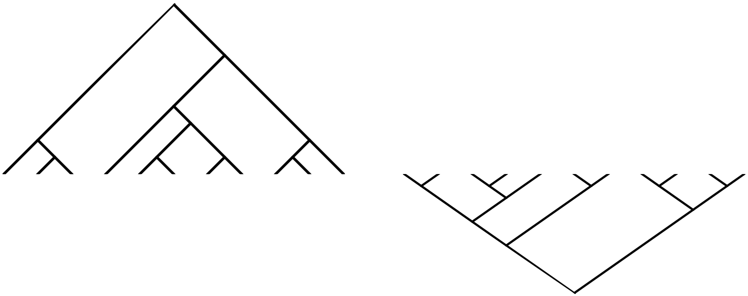

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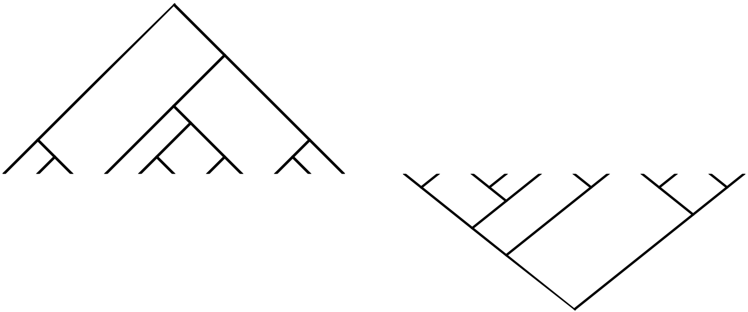

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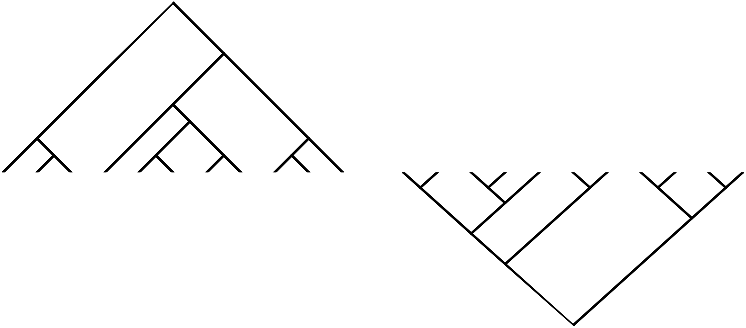

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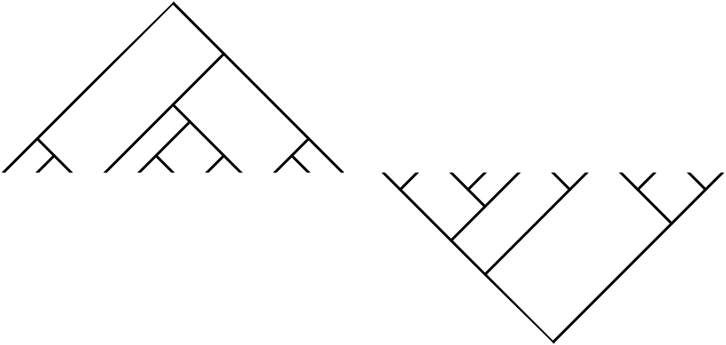

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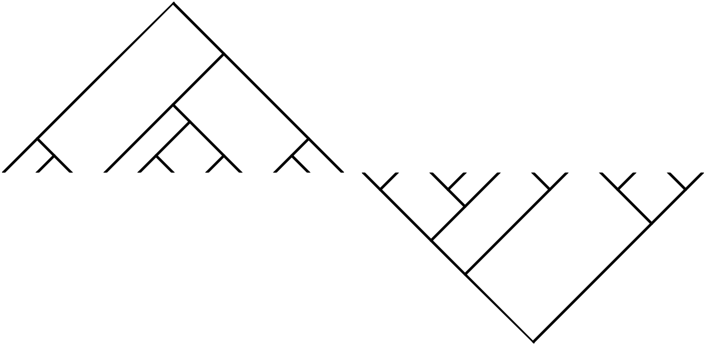

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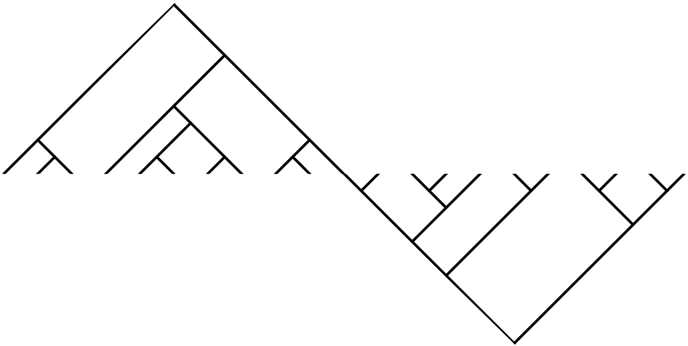

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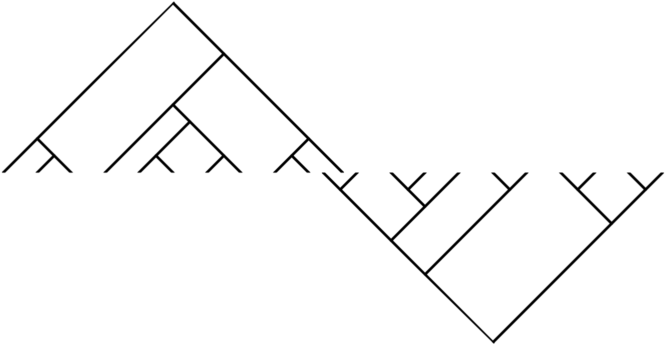

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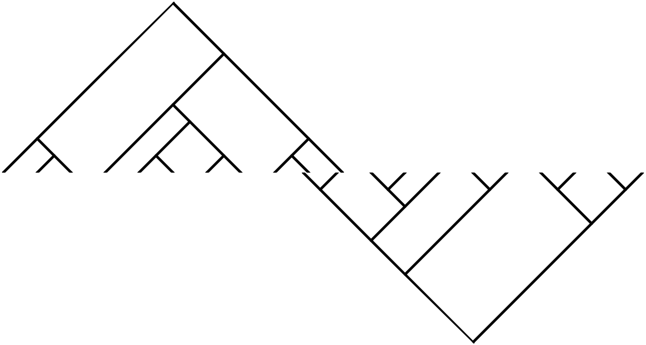

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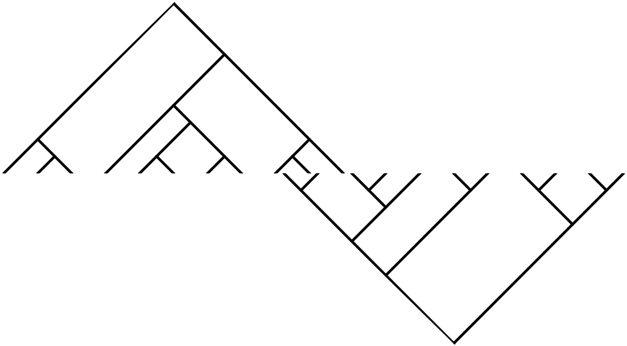

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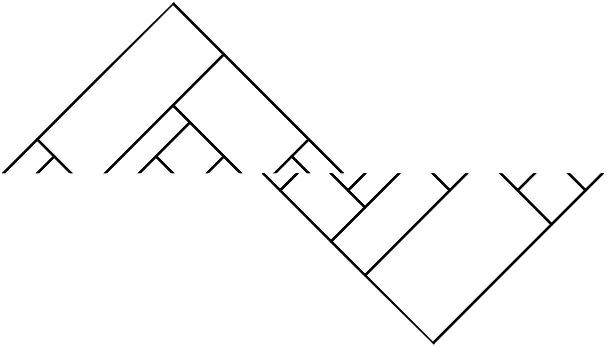

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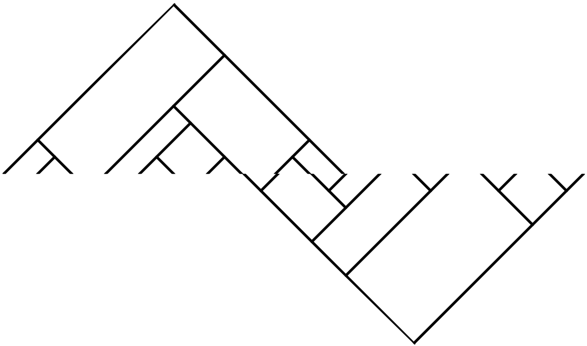

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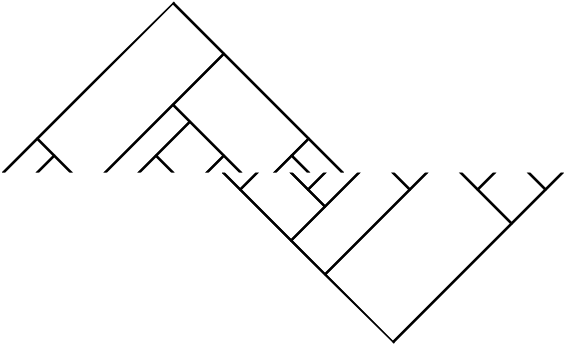

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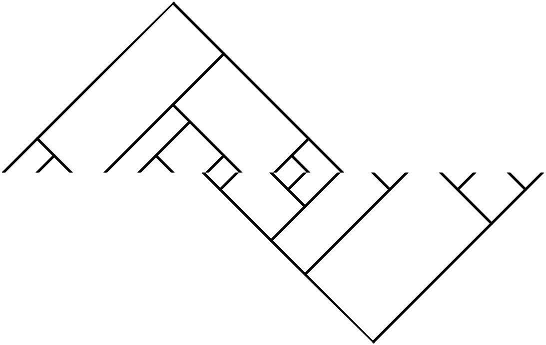

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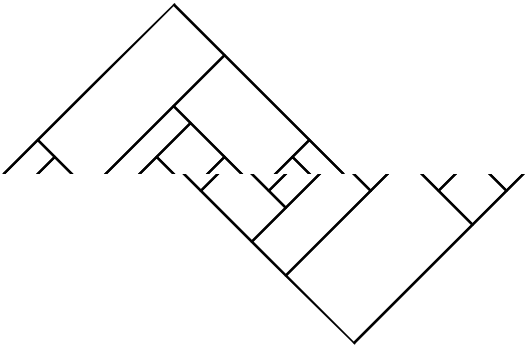

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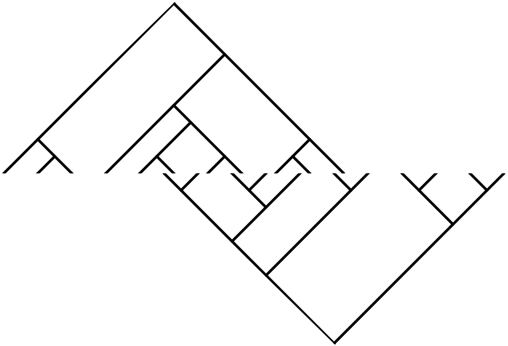

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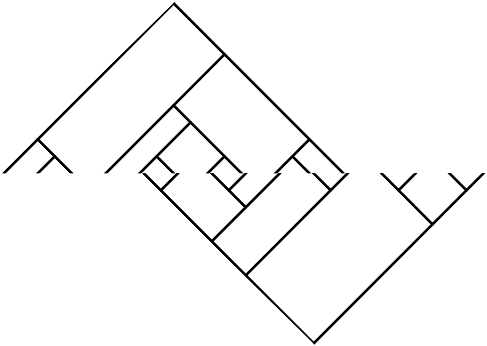

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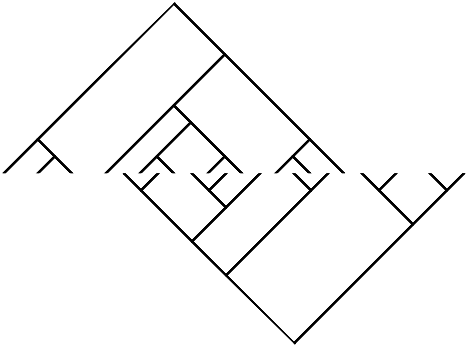

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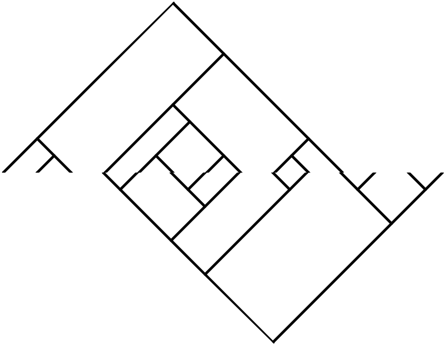

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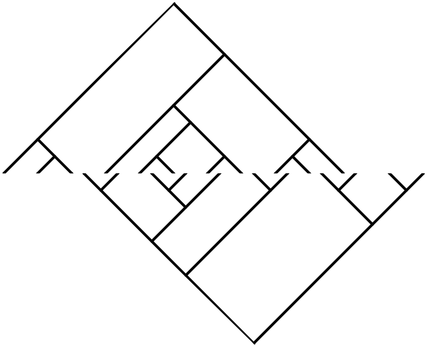

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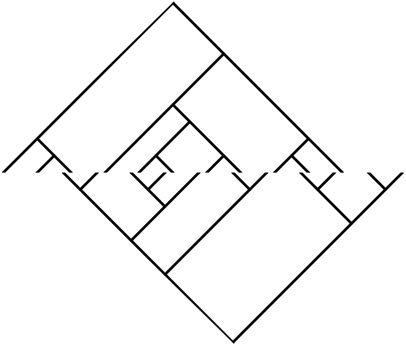

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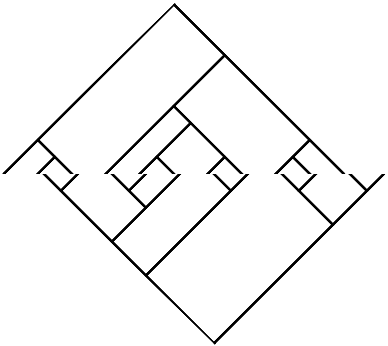

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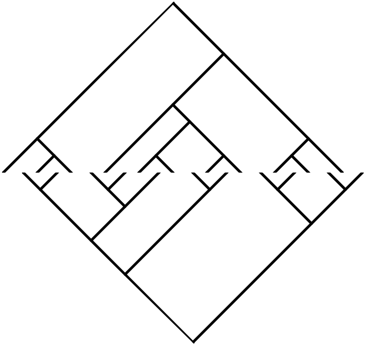

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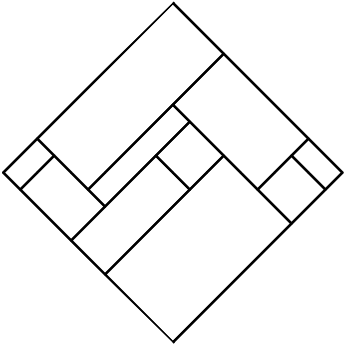

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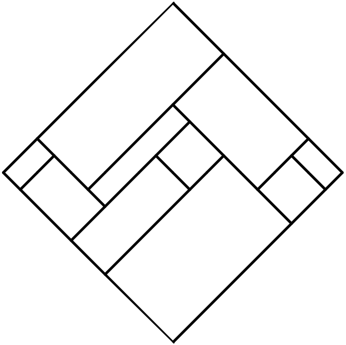

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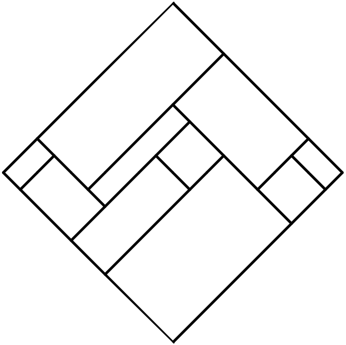

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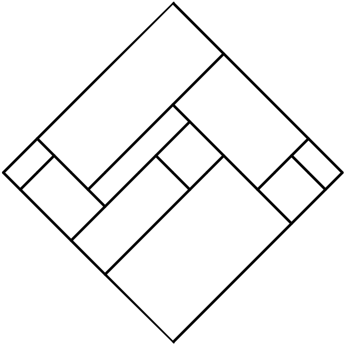

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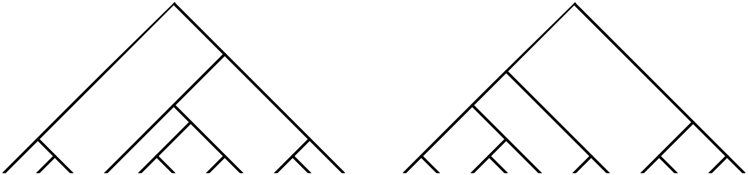
Dulucq-Guibert '96: Twin binary trees

- Counted by the Baxter numbers

Ackerman-Barequet-Pinter '06: (Diagonal) rectangulations.

- Counted by the Baxter numbers
- Gives a simple way to understand "twin" binary trees:



# How I like to think about this

$\eta_{312}$ : {permutations} $\rightarrow$ {planar binary trees}.

- Lattice homomorphism from weak order to Tamari lattice.
- Fibers (preimages of trees) define a lattice congruence $\Theta_{312}$.
- In particular, fibers are intervals.
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Two trees are twins iff they are $(\eta_{312}(x), \eta_{231}(x))$ for some $x$.

Fibers of $x \mapsto (\eta_{312}(x), \eta_{231}(x))$ are a congruence $\Theta_{312} \wedge \Theta_{231}$.
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This is not what the talk is about.

# Twin binary trees in a broader context

Given a Coxeter group $W$ and an orientation $c$ of the Coxeter diagram, construct the $c$-Cambrian congruence $\Theta_c$ on the weak order on $W$.

- congruence classes counted by the Catalan number.
- $W/\Theta_c$ is the $c$-Cambrian lattice.

Define the $c$-biCambrian congruence to be $\Theta_c \wedge \Theta_{c^{-1}}$.
($c^{-1}$ is the opposite orientation.)
In general, the number of classes of $\Theta_c \wedge \Theta_{c^{-1}}$ depends on the choice of $c$.

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**Example:** Type $A_n$

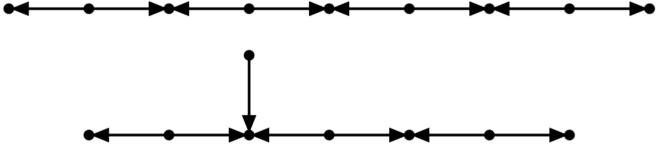
$c$ = 

$c^{-1}$ = 

Congruence classes of $\Theta_c \wedge \Theta_{c^{-1}}$ are counted by the Baxter number. (For other type-A cases, see Châtel-Pilaud 2014.)

# Bipartite biCambrian congruences

Some Coxeter diagrams are not paths. (No "linear" orientation!)
But there is a way to uniformly choose an orientation, because every Coxeter diagram is bipartite.



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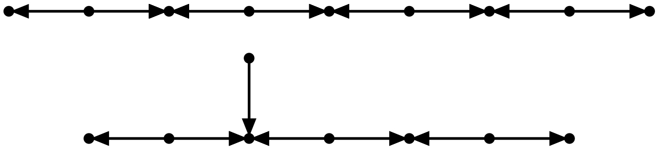

**Question:** How many congruence classes in $\Theta_c \wedge \Theta_{c^{-1}}$, for $c$ bipartite?

Is this a good question?

**Theorem** (Barnard 2014, in Barnard-R. 2016).
The bipartite biCambrian congruence has $\binom{2n}{n}$ classes in type $A_n$ and $2^{2n-1}$ classes in type $B_n$.

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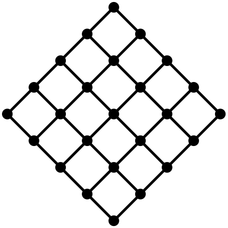

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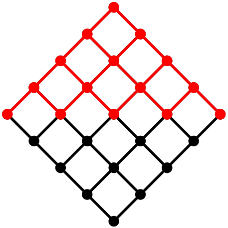

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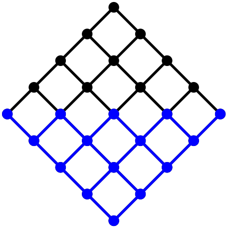

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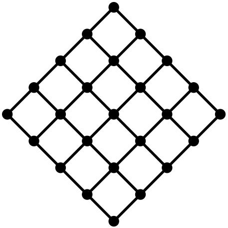

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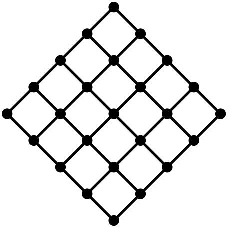
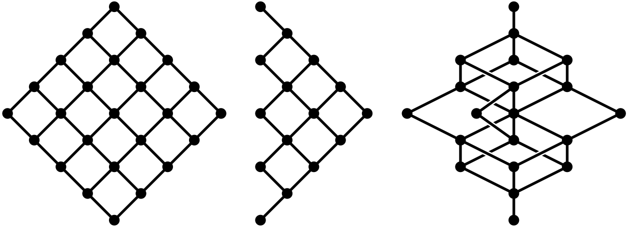


Hmm...

# Wishful speculation

**Definition.** The doubled root poset is two copies of the root poset—one upside-down—identified at the simple roots.



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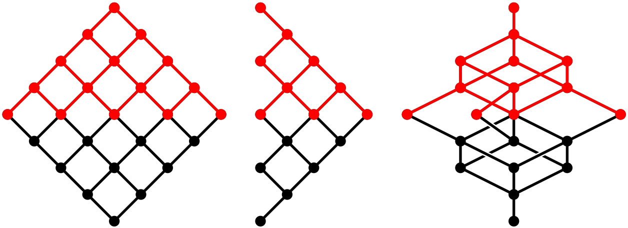

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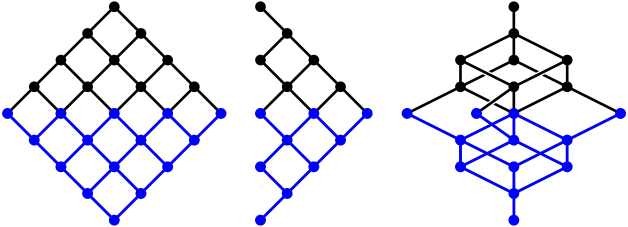

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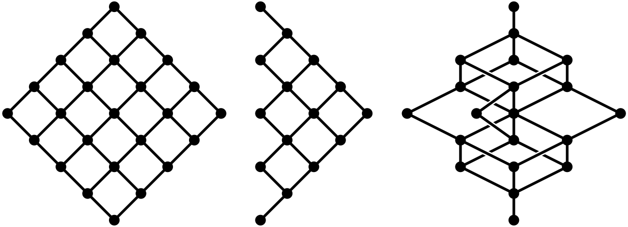

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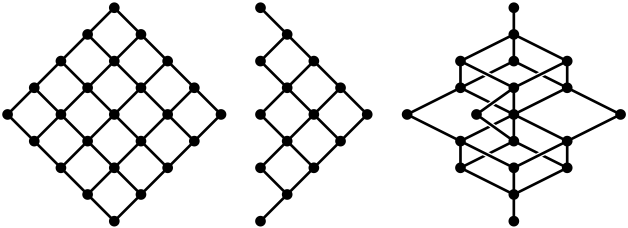

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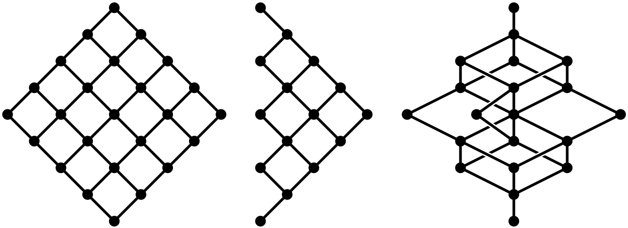

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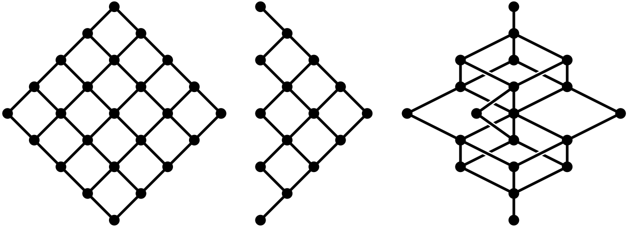

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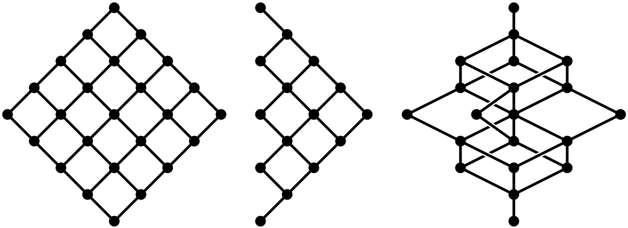

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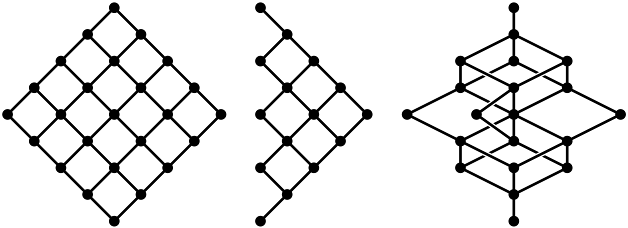

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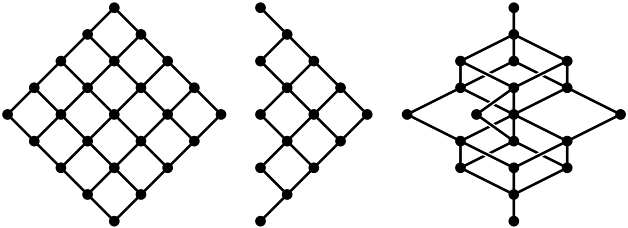

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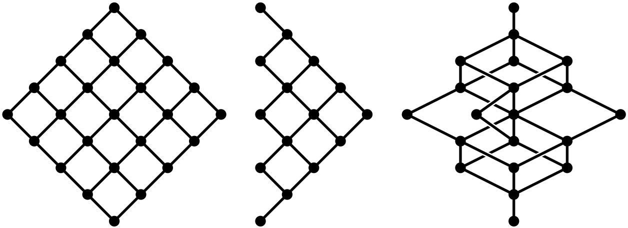

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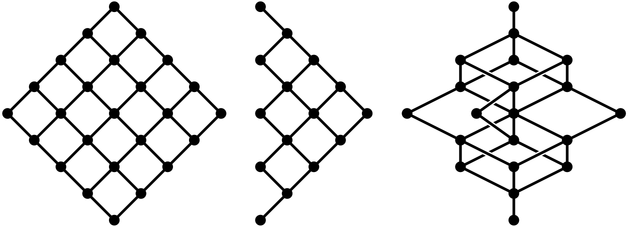

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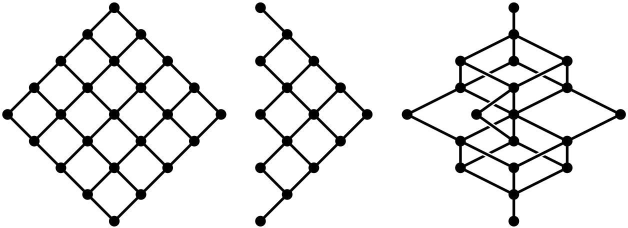

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# Coxeter-biCatalan combinatorics

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The Coxeter-biCatalan numbers:

| $W$ | $A_n$ | $B_n$ | $D_n$ |
|---|---|---|---|
| $\mathrm{biCat}(W)$ | $\binom{2n}{n}$ | $2^{2n-1}$ | $6 \cdot 4^{n-2} - 2\binom{2n-4}{n-2}$ |

| $E_6$ | $E_7$ | $E_8$ | $F_4$ | $H_3$ | $H_4$ | $I_2(m)$ |
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**Formula:** $\prod_{i=1}^{n} \frac{h+e_i-1}{e_i}$. Only works for $A_n$, $B_n$, $H_3$, $I_2(m)$

Ordinary Coxeter-Catalan combinatorics features, among other things:

- Noncrossing partitions,
- Nonnesting partitions (antichains in the root poset),
- clusters of almost pos. roots (e.g. $\Delta$-ations of a polygon),
- sortable elements.

In the Coxeter-biCatalan world, there are

- twin noncrossing partitions,
- twin nonnesting partitions (antichains in the doubled root poset),
- twin clusters,
- twin sortable elements,
- bisortable elements.

# Plan for the rest of the talk

- Details of the definitions
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In case time does not allow, a few brief comments on the proof:

• The Coxeter-biCatalan result depends on the analogous Coxeter-Catalan result, but not in a trivial way.

• Emily Barnard proved the type-A (and B) case in a way that provided a lot of insight and showed the way to a general proof. (Typically, type A proofs may not be so helpful for general-type proofs.)

# Orienting rank-two parabolics

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What is a rank-two parabolic root subsystem? Intersect the root system with a plane (and get a subset that spans the plane).

For the symmetric group $S_n$, roots are $e_i - e_j$ for $i \neq j$. Rank-two parabolic root subsystems are $\{\pm(e_i - e_j), \pm(e_i - e_k), \pm(e_j - e_k)\}$ for $i < j < k$..

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In the symmetric group, for $i < j < k$ we orient either
$e_i - e_j \to e_i - e_k \to e_j - e_k$ or $e_i - e_j \leftarrow e_i - e_k \leftarrow e_j - e_k$
depending on the sign of $\omega_c(e_i - e_j, e_j - e_k)$.

In fact, this depends only on whether $c$ has $s_{j-1} \to s_j$ or $s_{j-1} \leftarrow s_j$.

# $c$-aligned elements

For each rank-two parabolic root subsystem, the inversion set of $w \in W$ is "built up from one side or the other."

[*insert hand-waving here*]

An element $w \in W$ is $c$-aligned if, for every rank-two parabolic root subsystem, the inversion set of $w$ is built up in the direction given by the orientation of the rank-two parabolic root subsystem.

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In the symmetric group $S_n$:

Choosing an orientation $c$ corresponds to choosing a barring of each $i \in \{2, \dots, n-1\}$ as:

| | | |
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A permutation is $c$-aligned if it avoids the patterns $31\underline{2}$ and $\overline{2}31$.

Linear orientations: 312-avoiding or 231-avoiding permutations.

# Cambrian congruences

Weak order: Containment order on inversion sets.

For the symmetric group, you go down by a cover by undoing an inversion involving adjacent entries. For example $25341 \gtrdot 23541$.

Fact (nontrivial): Fix $c$. For each $w \in W$, there exists a unique (weak order) maximal $c$-aligned element $\pi_\downarrow^c(w)$ below $w$.

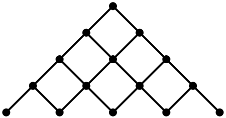
The $c$-Cambrian congruence $\Theta_c$ sets $v \equiv w$ iff $\pi_\downarrow^c(v) = \pi_\downarrow^c(w)$.

The bottom elements of $\Theta_c$ are the $c$-aligned elements.
(These coincide with the $c$-sortable elements.)

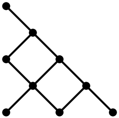
# Root poset

Partial order on the positive roots with $\alpha \leq \beta$ if and only if $\beta - \alpha$ is in the nonnegative span of the simple roots.

Example:



$S_6$



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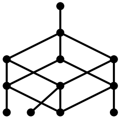

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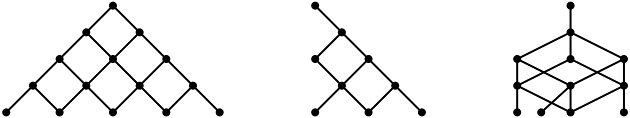


$S_6$ $B_3$ $D_4$

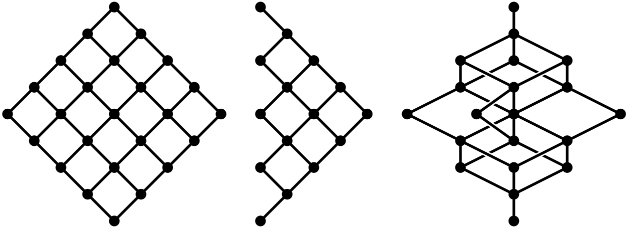
For fixed $c$,

$$\# \text{ } c\text{-aligned elements} = \# \text{ } \Theta_c\text{-classes}$$
$$= \# \text{ antichains in the root poset}$$

- This number is the $W$-Catalan number.
- First equality: by definition.
- Second: central mystery of Coxeter-Catalan combinatorics.

# Doubled root poset and biCambrian congruences

**Definition.** The doubled root poset is two copies of the root poset—one upside-down—identified at the simple roots.



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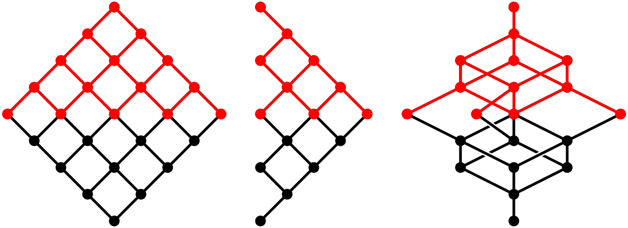

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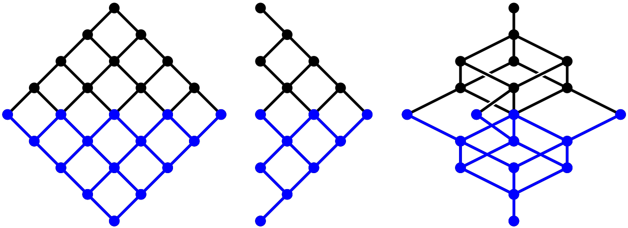

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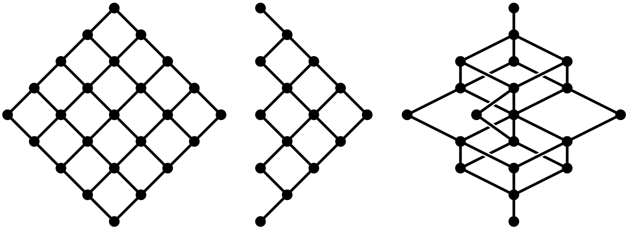

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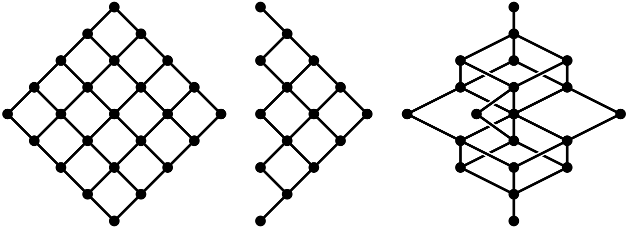

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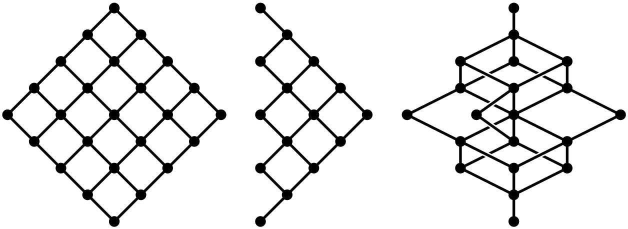
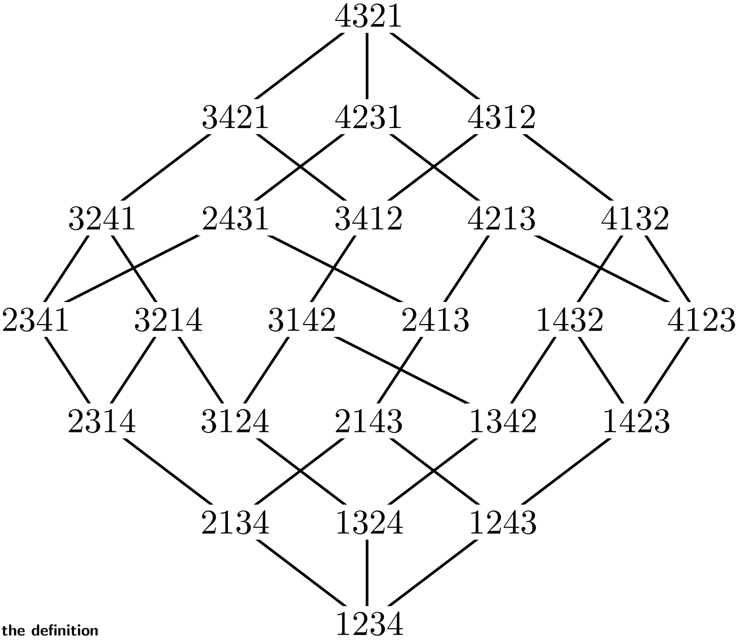


**Definition.** The $c$-biCambrian congruence is $\Theta_c \wedge \Theta_{c^{-1}}$.

**Theorem** (Barnard-R. 2015)

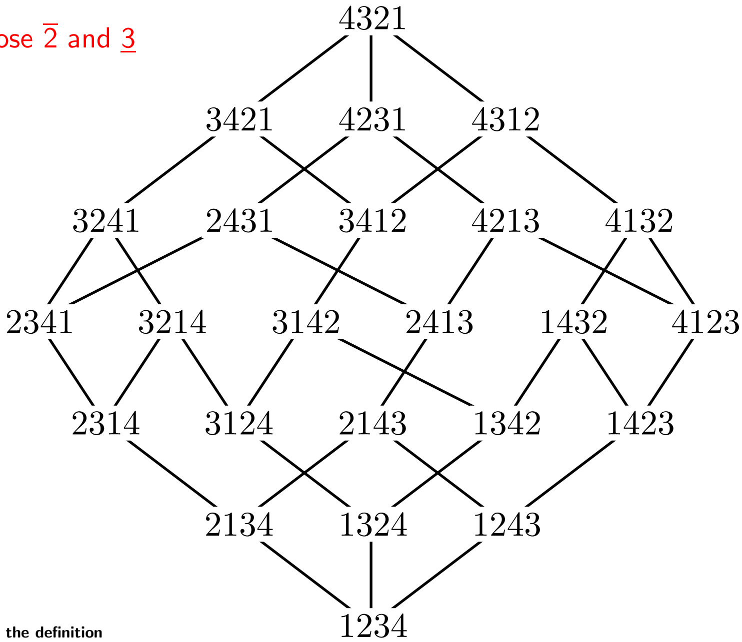
$$\#\text{ antichains in the doubled root poset} = \#\text{ classes in the bipartite biCambrian congruence}$$

# Example: $W = S_4$ (type $A_3$)



# Example: $W = S_4$ (type $A_3$)

Choose $\overline{2}$ and $\underline{3}$



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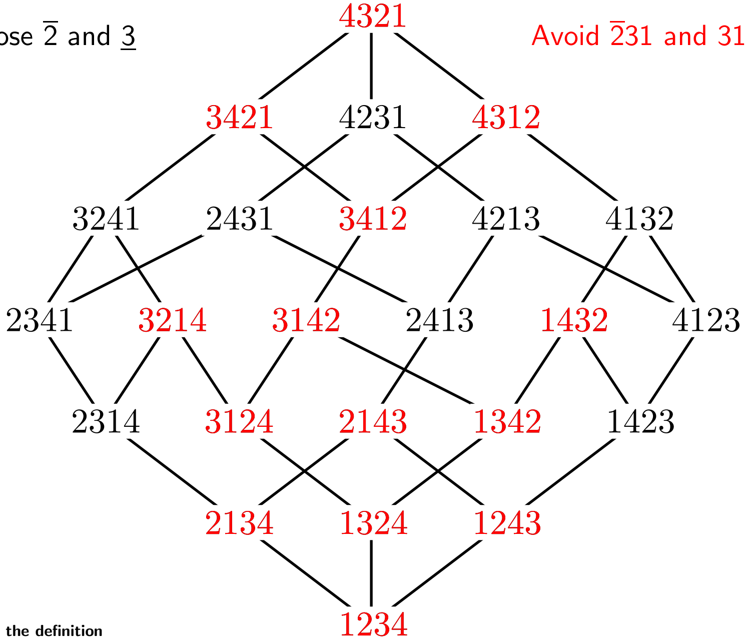
Choose $\overline{2}$ and $\underline{3}$

Avoid $\overline{2}31$ and $31\underline{2}$



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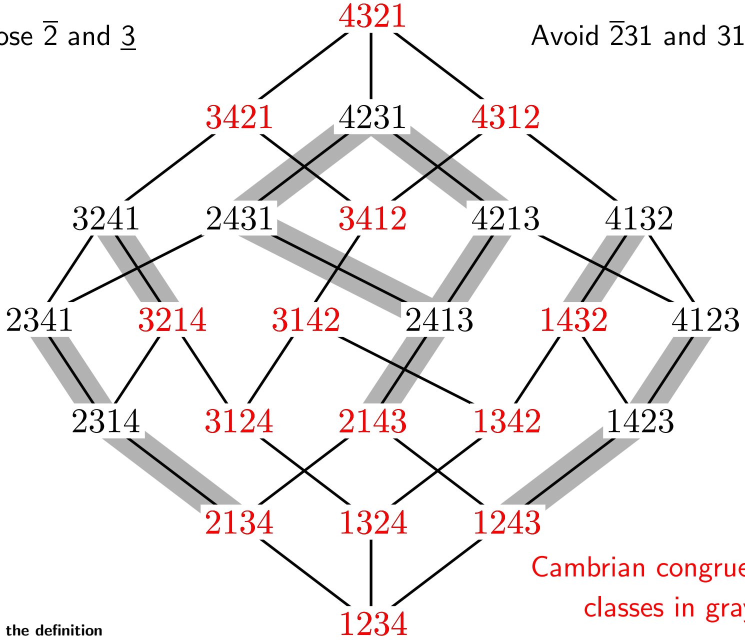


Cambrian congruence classes in gray

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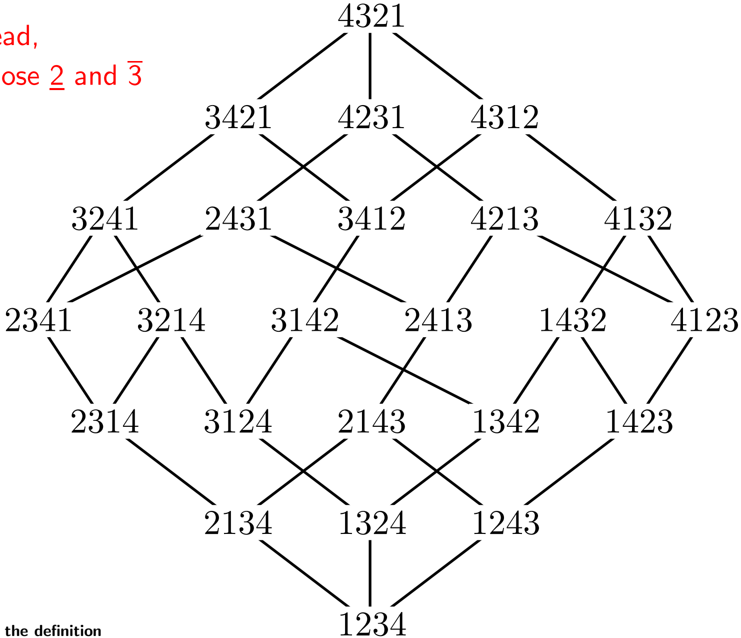
Instead,
choose $\underline{2}$ and $\overline{3}$



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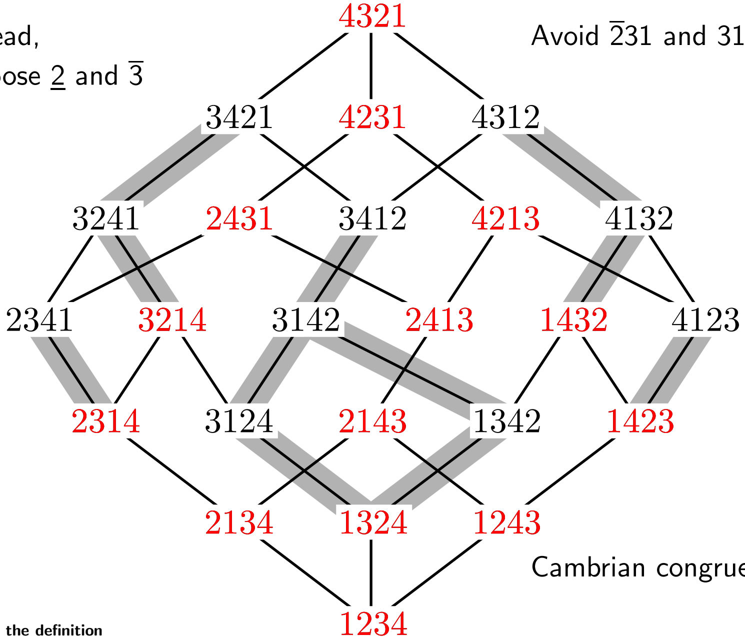


Cambrian congruence

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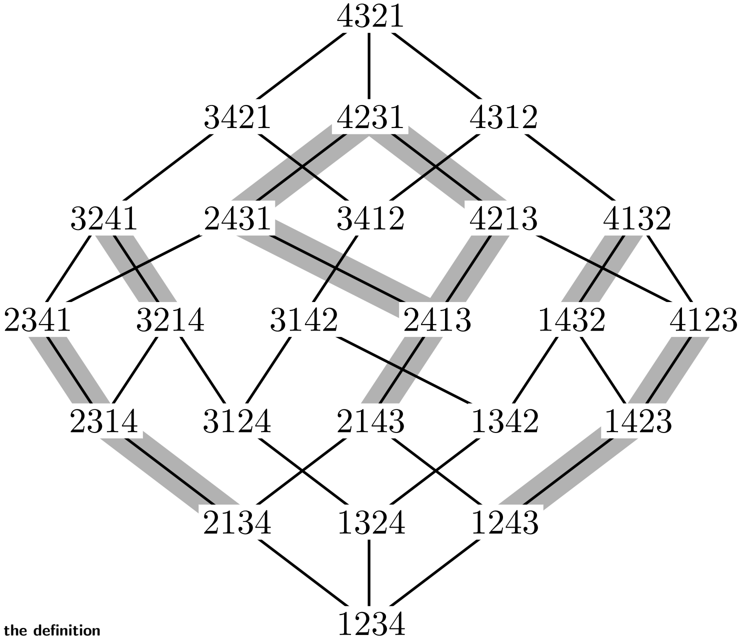

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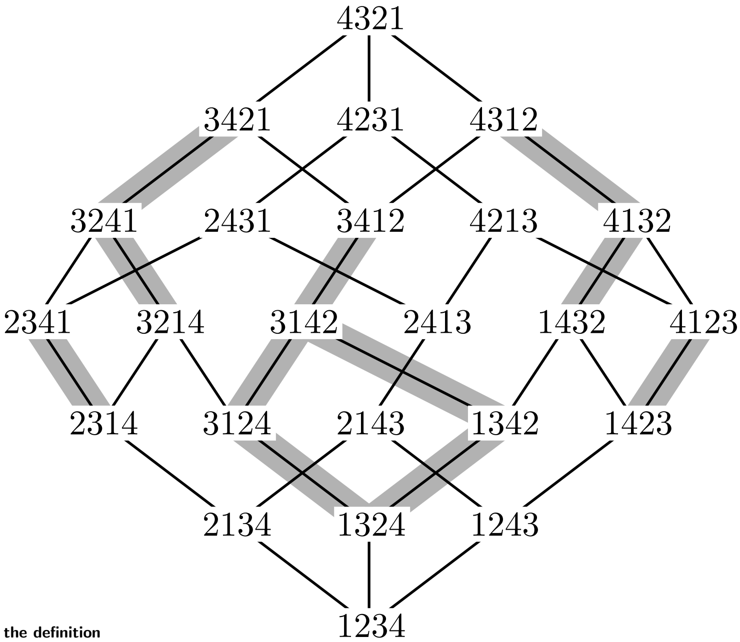
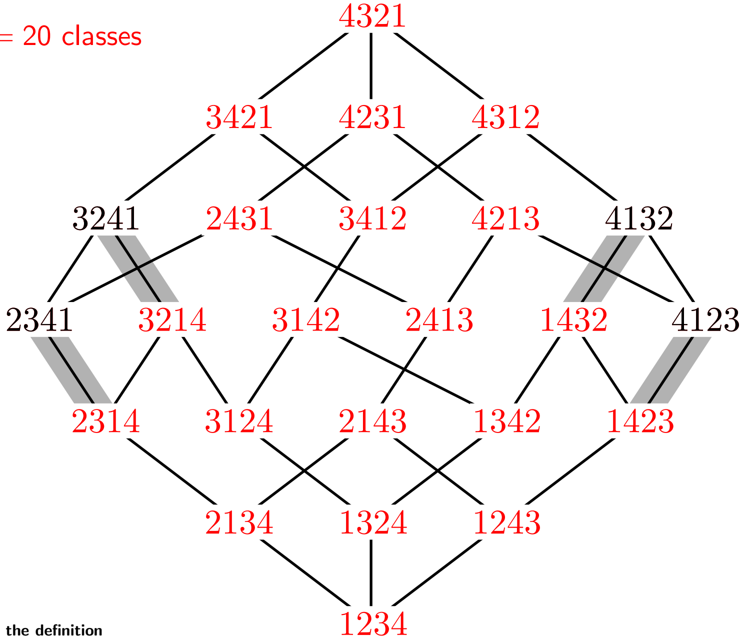


# Example: $W = S_4$ (type $A_3$)

$\binom{6}{3} = 20$ classes



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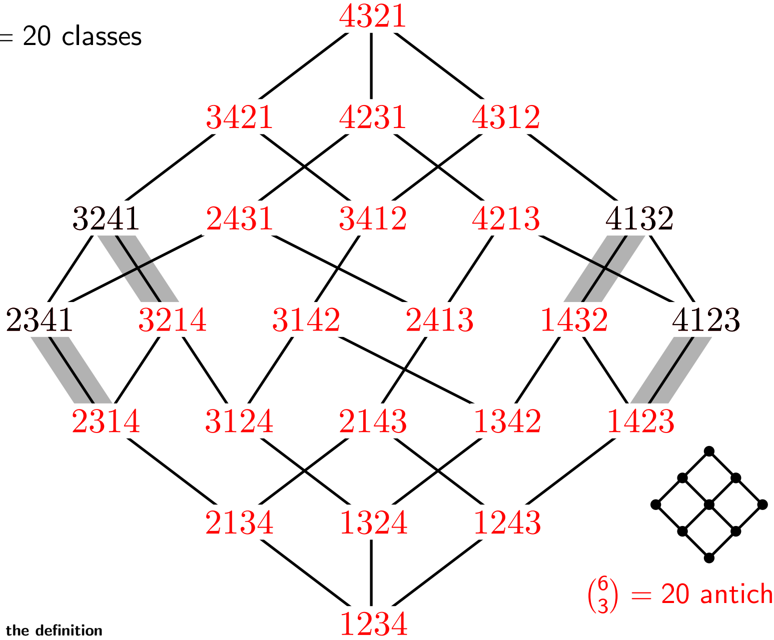



$\binom{6}{3} = 20$ antichains

# Proof idea: Double-positive Catalan numbers

**Proposition.** The number of antichains in the root poset for $W$ with full support is

$$\mathrm{Cat}^{+}(W) = \sum_{J \subseteq S} (-1)^{|S|-|J|} \,\mathrm{Cat}(W_J). \tag{1}$$

**Proposition.** The number of antichains in the root poset for $W$ with full support containing no simple roots is

$$\mathrm{Cat}^{++}(W) = \sum_{J \subseteq S} (-1)^{|S|-|J|} \,\mathrm{Cat}^{+}(W_J). \tag{2}$$

**Theorem.** For any finite Coxeter group $W$ with simple generators $S$, the number of antichains in the doubled root poset is

$$\sum 2^{|S \setminus (I \cup J)|} \,\mathrm{Cat}^{++}(W_I)\,\mathrm{Cat}^{++}(W_J),$$

where the sum is over all ordered pairs $(I, J)$ of *pairwise disjoint* subsets of $S$.

# Proof idea: Double-positive Catalan numbers (continued)

**Definition.** The bottom elements of the $c$-biCambrian congruence classes are called $c$-bisortable elements.

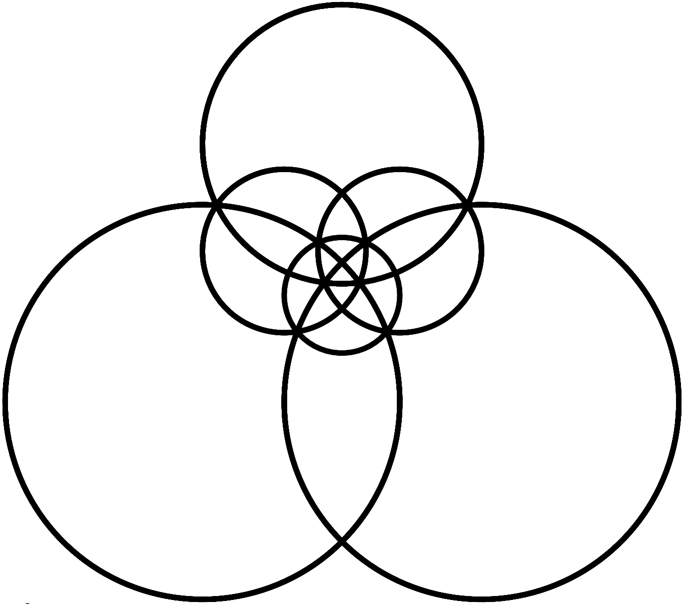
The proof concludes by showing that for bipartite $c$ only, the $c$-bisortable elements are also counted by

$$\sum 2^{|S \setminus (I \cup J)|} \operatorname{Cat}^{++}(W_I) \operatorname{Cat}^{++}(W_J),$$

This involves fun lattice theory like canonical join representations.

It also involves the combinatorics of $c$-sortable and $c^{-1}$-sortable elements.

# Bipartite biCambrian fans



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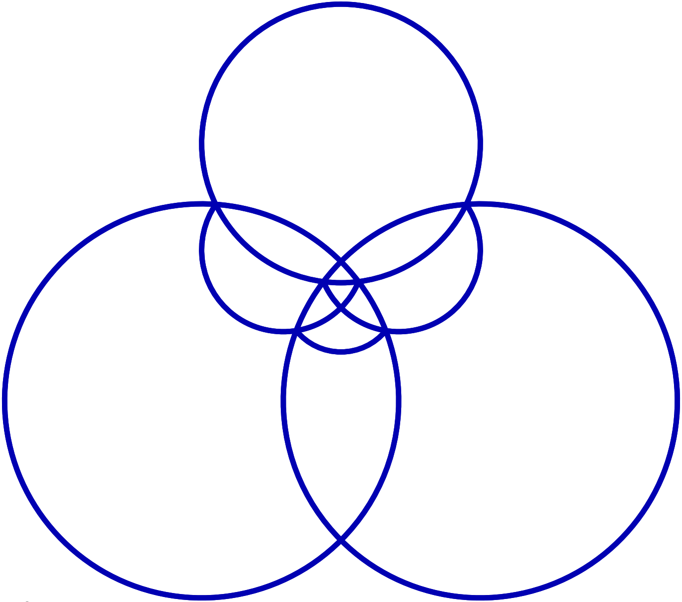

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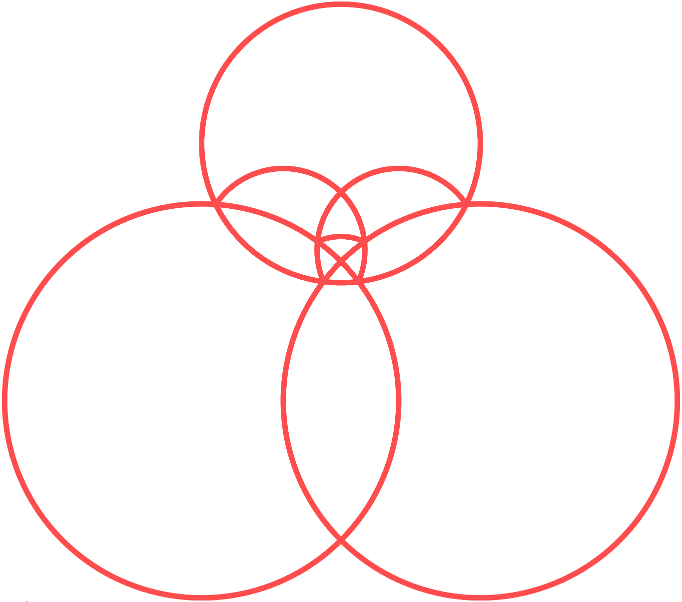

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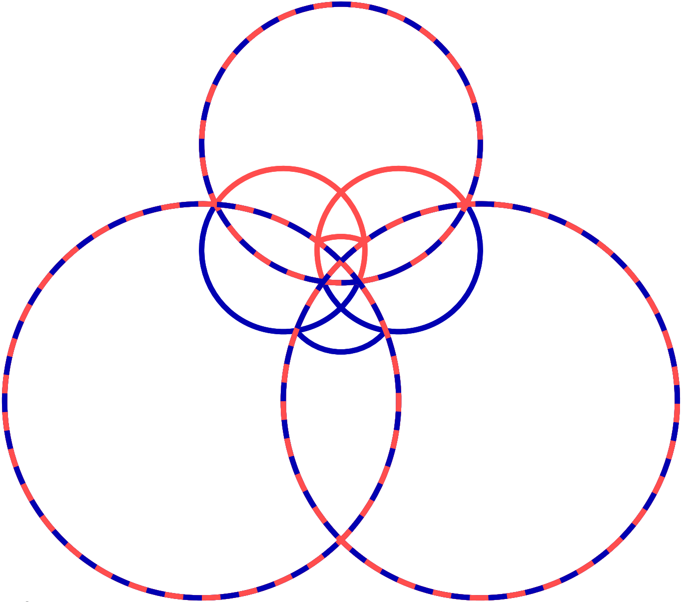
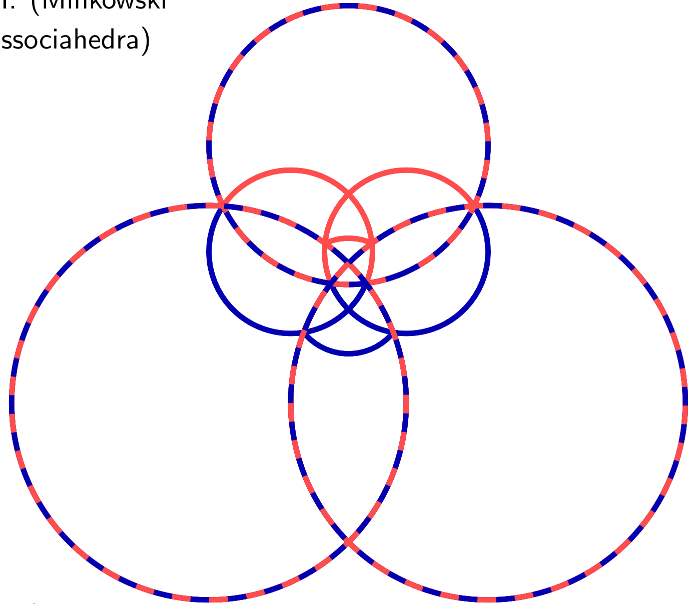


# Bipartite biCambrian fans

Polytopal. (Minkowski sum of associahedra)



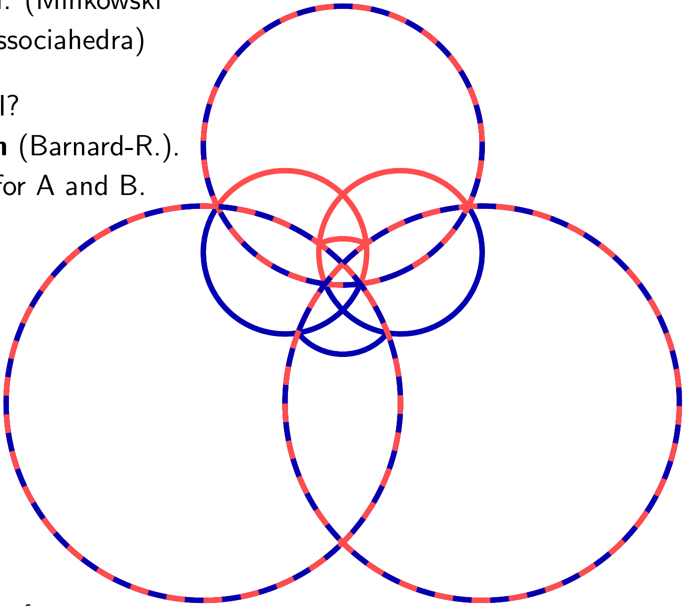
# Bipartite biCambrian fans

Polytopal. (Minkowski sum of associahedra)

Simplicial?

**Theorem** (Barnard-R.).

Yes for A and B.



Thanks for listening.