General linear groups as reflection group "wannabes"

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Algebraic combinatorics and group actions

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OUTLINE:

3 reflection group counting stories where GL_n wants in on the game...

① Cycling subsets

② q-Catalan numbers

③ reflection factorizations

① Cycling subsets

THM (R.-Stanton-White 2004)

When $\mathfrak{S}_n$ permutes k-element subsets of $\{1,2,\ldots,n\}$, the number fixed by the d^{th} power c^d of an n-cycle or (n-1)-cycle c is

$$\left.\begin{bmatrix} n \\ k \end{bmatrix}_q \right|_{\substack{q=\left(e^{\frac{2\pi i}{n}}\right)^d \\ \text{or } q=\left(e^{\frac{2\pi i}{n-1}}\right)^d}}$$

where...

$\begin{bmatrix} n \\ k \end{bmatrix}_q :=$ q-binomial coefficient

$$= \frac{[n]!_q}{[k]!_q \, [n-k]!_q}$$

with $[n]!_q := [n]_q [n-1]_q \cdots [2]_q [1]_q$

$$[n]_q = 1+q+q^2+\dots+q^{n-1} = \frac{1-q^n}{1-q}$$

EXAMPLE:

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \frac{[4]!_q}{[2]!_q \, [2]!_q} = \frac{[4]_q [3]_q}{[2]_q [1]_q}$$

$$= (1+q^2)(1+q+q^2) = 1+q+2q^2+q^3+q^4$$

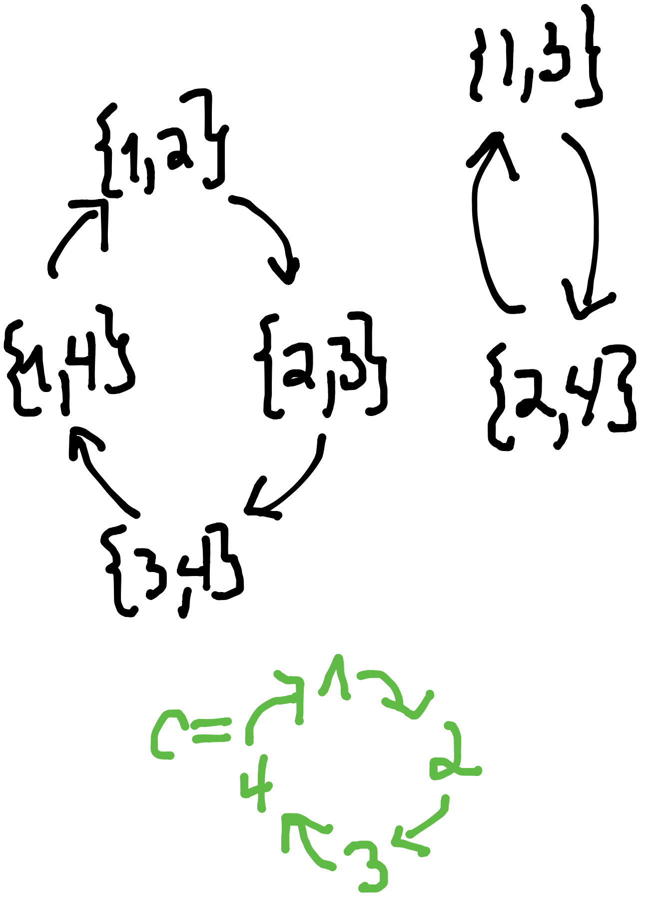
$$\begin{bmatrix}4\\2\end{bmatrix}_q = 1+q+2q^2+q^3+q^4$$

$q=\pm i$ → 0

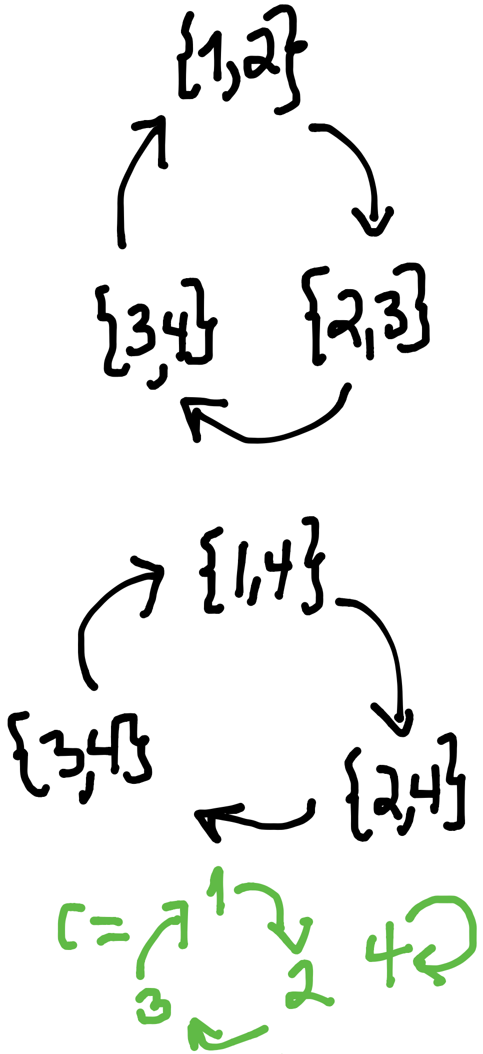
$q=-1$ → 2

$q=1$

$q=e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}$



n-cycle



(n-1)-cycle

THM (R-Stanton-White 2004)

When $GL_n(\mathbb{F}_q)$ permutes k-dimensional subspaces of $\mathbb{F}_q^n$, the number fixed by the d^{th} power c^d of a Singer cycle c is

(any multiplicative generator for $\mathbb{F}_{q^n}^\times \hookrightarrow GL_{\mathbb{F}_q}(\mathbb{F}_{q^n}) \cong GL_n(\mathbb{F}_q)$)

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q,t} \Bigg|_{t=\left(e^{\frac{2\pi i}{q^n-1}}\right)^d}$$

where...

$\begin{bmatrix} n \\ k \end{bmatrix}_{q,t} :=$ (q,t)-binomial coefficient

$$= \frac{n!_{q,t}}{k!_{q,t}\ (n-k)_{q,t^{q^k}}}$$

where $n!_{q,t} :=$

$$(1-t^{q^n-q^0})(1-t^{q^n-q^1})\cdots(1-t^{q^n-q^{n-1}})$$

$$\begin{bmatrix}4\\2\end{bmatrix}_{q=2,t} = \frac{4!_{2,t}}{2!_{2,t}\cdot 2!_{2,t^{2^2}}}$$

$$= \frac{\left(1-t^{2^4-2^0}\right)\left(1-t^{2^4-2^1}\right)}{\left(1-t^{2^2-2^0}\right)\left(1-t^{2^2-2^1}\right)}$$

$$= \frac{(1-t^{15})(1-t^{14})}{(1-t^{3})(1-t^{2})}$$

$$= (1+t^3+t^6+t^9+t^{12})(1+t^2+t^4+t^6+t^8+t^{10}+t^{12})$$

Where do reflection groups play any role in the above?

First let's define them...

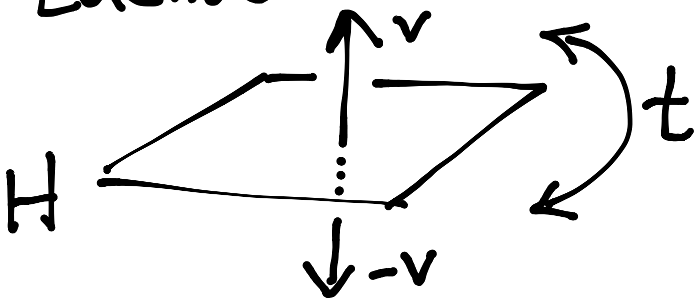
DEF'N:

A reflection t in $GL_n(\mathbb{F})$ is an element whose fixed subspace $(\mathbb{F}^n)^t = \ker(t-1)$ is a hyperplane H

codimension 1 linear subspace

EXAMPLES:

- Euclidean reflections



- Unitary reflections

$$t = \begin{bmatrix} e^{\frac{2\pi i}{d}} & & & & \\ & 1 & & & 0 \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{bmatrix}$$

- Transvections

$$t = \begin{bmatrix} \begin{array}{cc|} 1 & 1 \\ 0 & 1 \\ \hline \end{array} & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ 0 & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

- Infinite order is OK!

$$t = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \in GL_2(\mathbb{R})$$

DEF'N: A subgroup $G \leq GL_n(F)$ is a reflection group if it is generated by reflections.

However...

DEF'N: A subgroup $G \leq GL_n(F)$ is a finite reflection group if it is finite, and when acting on the polynomials

$$S := \mathbb{F}[x_1, \dots, x_n]$$

its G-invariant subalgebra S^G is again polynomial

$$S^G = \mathbb{F}[f_1, f_2, \dots, f_n]$$

REMARKS:

- **THM:** (Serre 1967)
  Finite $G \leq GL_n(\mathbb{F})$ with S^G polynomial are necessarily generated by reflections.

- **THM:** (Chevalley, 1955 Shephard-Todd)
  Finite $G \leq GL_n(\mathbb{F})$ with $char(\mathbb{F})=0$ have S^G polynomial
  $\iff$ G gen'd by reflections.

THM (R-Stanton-White 2004
Broer-R-Smith-Webb 2011)

When a finite reflection group W transitively permutes a set $X = W/W'$, any $w \in W$ of order m which is regular (in Springer's 1972 sense)

w has an eigenvector in $\mathbb{F}_q^n$ fixed by no reflections of W

fixes
$$\left.\frac{\mathrm{Hilb}(S^{W'},t)}{\mathrm{Hilb}(S^{W},t)}\right|_{t=e^{\frac{2\pi i}{m}}}$$
elements of X.

DEFN: A graded $\mathbb{F}$-vectorspace $R = \bigoplus_{d=0}^{\infty} R_d$ has Hilbert series

$$\mathrm{Hilb}(R,t) := \sum_{d=0}^{\infty} \dim_{\mathbb{F}} R_d \cdot t^d$$

EXAMPLE

$$\mathfrak{S}_n \circlearrowright S = \mathbb{F}[x_1, \ldots, x_n]$$

$$S^{\mathfrak{S}_n} = \mathbb{F}[e_1, e_2, \ldots, e_n]$$

where $\prod_{i=1}^{n}(t+x_i) = t^n + e_1 t^{n-1} + \ldots + e_{n-1} t + e_n$

$\Longrightarrow$ degree of e_i is i, and

$$\mathrm{Hilb}(S^{\mathfrak{S}_n}, t) = \prod_{i=1}^{n} \frac{1}{1-t^i}$$

Meanwhile, $W = \mathfrak{S}_n$ permutes

$$X = \{k\text{-subsets of } \{1,2,\ldots,n\}\}$$

$$= \mathfrak{S}_n / \mathfrak{S}_k \times \mathfrak{S}_{n-k} = W/W'$$

with

$$S^{W'} = \mathbb{F}[e_1(x_1,\ldots,x_k),\ldots,e_k(x_1,\ldots,x_k), e_1(x_{k+1},\ldots,x_n),\ldots,e_{n-k}(x_{k+1},\ldots,x)]$$

$\Rightarrow$

$$\frac{\mathrm{Hilb}(S^{W'},t)}{\mathrm{Hilb}(S^{W},t)} = \frac{(1-t^1)(1-t^2)\cdots(1-t^n)}{(1-t^1)\cdots(1-t^k)\cdot(1-t^1)\cdots(1-t^{n-k})}$$

$$= \begin{bmatrix} n \\ k \end{bmatrix}_t$$

Furthermore, the Springer regular elements w inside $W=\mathfrak{S}_n$ are exactly the powers of n-cycles and $(n-1)$-cycles:

$c=(1,2,\ldots,n)$ has eigenvector $(1,\zeta,\zeta^2,\ldots,\zeta^{n-1})$ if $\zeta=e^{\frac{2\pi i}{n}}$ avoiding all hyperplanes $x_i=x_j$

$c=(1,2,\ldots,n-1)(n)$ has eigenvector $(1,\zeta,\zeta^2,\ldots,\zeta^{n-2},0)$ if $\zeta=e^{\frac{2\pi i}{n-1}}$ similarly avoiding all $x_i=x_j$

Certainly $GL_n(\mathbb{F}_q)$ is finite, but is it a finite reflection group? Yes.

THM (L.E. Dickson 1911):

$S=\mathbb{F}_q[x_1,\ldots,x_n]$ has

$S^{GL_n(\mathbb{F}_q)}=\mathbb{F}_q[f_1,f_2,\ldots,f_n]$ where

$$\prod_{\substack{(c_1,\ldots,c_n)\\ \in\mathbb{F}_q^n}}\left(t+(c_1x_1+\ldots+c_nx_n)\right)=t^{q^n}+f_1(\underline{x})t^{q^{n-1}}+f_2(\underline{x})t^{q^{n-2}}+\ldots+f_{n-1}(\underline{x})t^{q^1}+f_n(\underline{x})t^{q^0}$$

In particular, degree of f_i is q^n-q^{n-i}

so $\mathrm{Hilb}(S^{GL_n(\mathbb{F}_q)},t)=\dfrac{1}{n!_{q,t}}$

Meanwhile, $GL_n(\mathbb{F}_q)$ permutes

$X =$ k-dimensional subspaces of $\mathbb{F}_q^n$

$$= GL_n(\mathbb{F}_q) / P_k$$

where $P_k = \left\{ \left[\begin{array}{c|c} \overbrace{\quad * \quad}^{k} & \overbrace{\quad * \quad}^{n-k} \\ \hline 0 & * \end{array} \right] \right\}$

and

THM (Kuhn and Mitchell 1984)

$$\frac{Hilb(S^{P_k},t)}{Hilb(S^{GL_n(\mathbb{F}_q)},t)} = \frac{n!_{q,t}}{k!_{q,t}\,(n-k)!_{q,t^{q^k}}}$$

$$= \begin{bmatrix} n \\ k \end{bmatrix}_{q,t}$$

Who are the Springer regular elements w inside $W = GL_n(\mathbb{F}_q)$?

That is, which w have an $\mathbb{F}_q^n$ eigenvector avoiding all $\mathbb{F}_q$-hyperplanes?

PROP: (R.-Stanton-Webb)

They are exactly the powers of Singer cycles, that is, elements of $\mathbb{F}_{q^n}^\times$ embedded inside $GL_n(\mathbb{F}_q)$.

COR:

- $W = \mathfrak{S}_n \curvearrowright \{k\text{-subsets}\}$,

  $w = c^d$ for c an m-cycle, $m = \begin{cases} n \\ \text{or} \\ n-1 \end{cases}$

  $$\Rightarrow w \text{ fixes } \left.\begin{bmatrix} n \\ k \end{bmatrix}_q \right|_{q=\left(e^{\frac{2\pi i}{n}}\right)^d}$$

  k-subsets

- $W = GL_n(\mathbb{F}_q) \curvearrowright \{k\text{-subspaces}\}$,

  $w = c^d$ for c a Singer cycle,

  $$\Rightarrow w \text{ fixes } \left.\begin{bmatrix} n \\ k \end{bmatrix}_{q,t} \right|_{t=\left(e^{\frac{2\pi i}{q^n-1}}\right)^d}$$

  k-subspaces

REMARK:

$GL_n(\mathbb{F}_q)$ is already behaving here more like the real reflection groups

- $W = W(B_n) = \mathfrak{S}_n^{\pm}$
  $=$ hyperoctahedral group of all $n \times n$ signed permutations

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

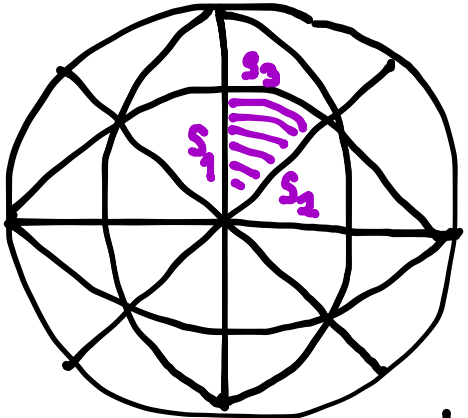
- $W = W(D_n)$
  $=$ its index 2 subgroup with evenly many -1's

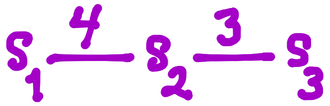
which also have

$$\left\{ \begin{matrix} \text{Springer's} \\ \text{regular elements} \end{matrix} \right\} = \left\{ \begin{matrix} \text{powers of} \\ \text{Coxeter elements} \end{matrix} \right\}$$

What's a Coxeter element c in a finite reflection group $W \leq GL_n(\mathbb{R})$?

- c conjugate to $s_1 s_2 \cdots s_n$ where $W = \langle s_1, \ldots, s_n \mid (s_i s_j)^{m_{ij}} \rangle$





- a regular element c with eigenvalue $e^{2\pi i/h}$ where

$h :=$ Coxeter number $=$ max degree $d_i = \deg(f_i)$

if $S^W = \mathbb{F}[f_1, f_2, \ldots, f_n]$

THESIS:

$GL_n(\mathbb{F}_q)$ thinks it is a real reflection group with

- Coxeter number $h = q^n - 1$.
- Coxeter elements = Singer cycles

e.g.

$$S^{GL_n(\mathbb{F}_q)} = \mathbb{F}_q[f_1, \ldots, f_{n-1}, f_n]$$

with degrees $q^n - q^{n-1}, \ldots, q^n - q, \; q^n - 1$

$$\begin{aligned} &\shortparallel \\ &h \\ &\shortparallel \end{aligned}$$

order of all Singer cycles

② q-Catalan numbers

Recall Catalan numbers

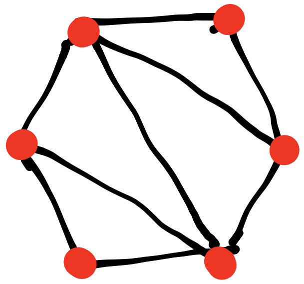
$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)(2n-1)\cdots(n+2)}{n(n-1)\cdots 2}$$

count many things, including triangulations of an $(n+2)$-gon

EXAMPLE $n=4$

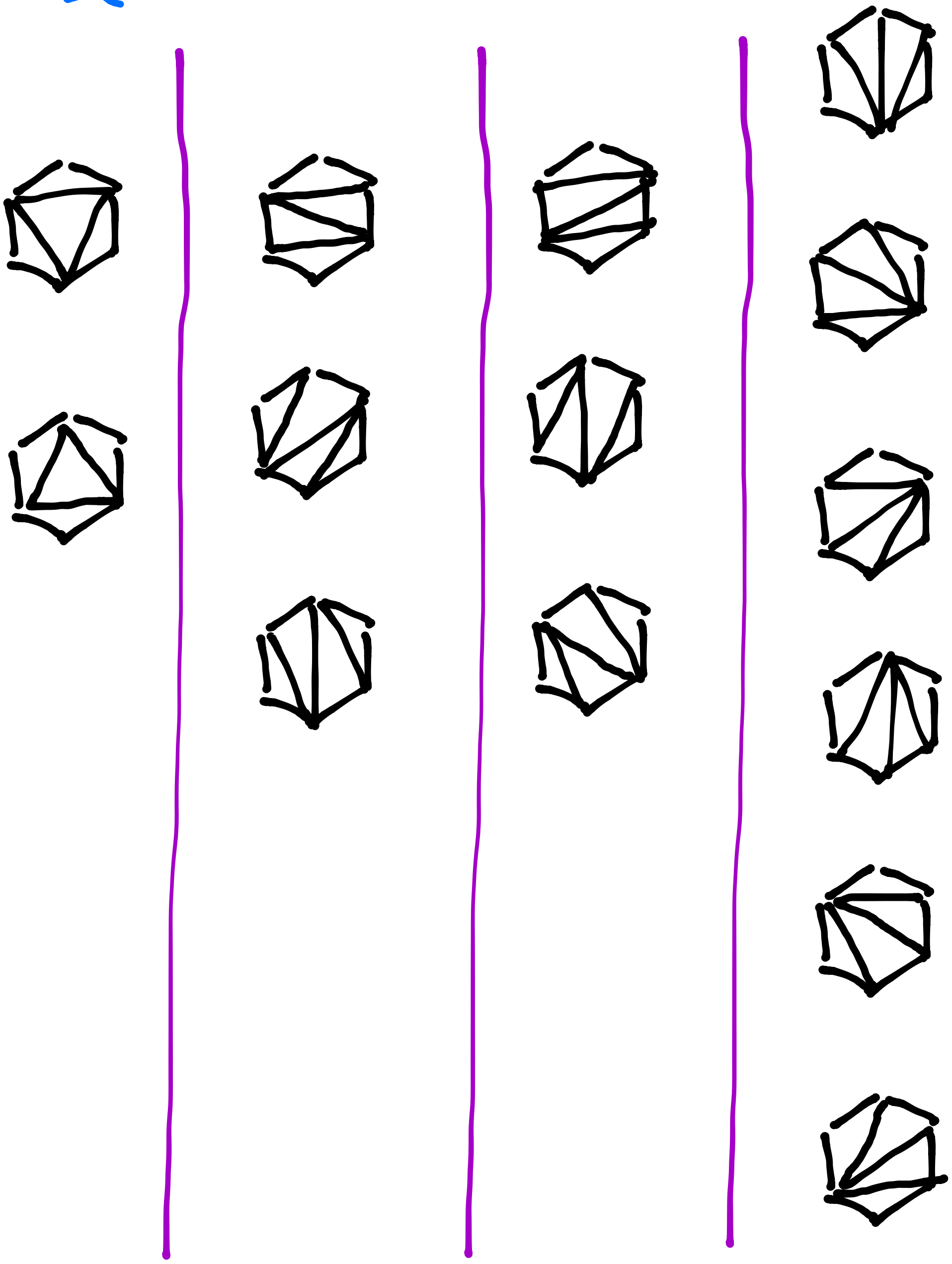
$$C_4 = \frac{1}{5}\binom{8}{4} = \frac{8\cdot7\cdot6}{4\cdot3\cdot2} = 14$$

counts these:



$C_4 = 14$

$= 2 + 3 + 3 + 6$



THM (R.-Stanton-White)

MacMahon's q-Catalan number

$$C_n(q) := \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$$

specialized to $q = \left(e^{\frac{2\pi i}{n+2}}\right)^d$

counts the triangulations

having $\frac{n+2}{d}$-fold symmetry.

EXAMPLE: $C_4(q) = \frac{1}{[5]_q} \begin{bmatrix} 8 \\ 4 \end{bmatrix}_q = \frac{[8]_q [7]_q [6]_q}{[4]_q [3]_q [2]_q}$

$$= 1+q^2+q^3+2q^4+q^5+2q^6+q^7+2q^8+q^9+q^{10}+q^{12}$$

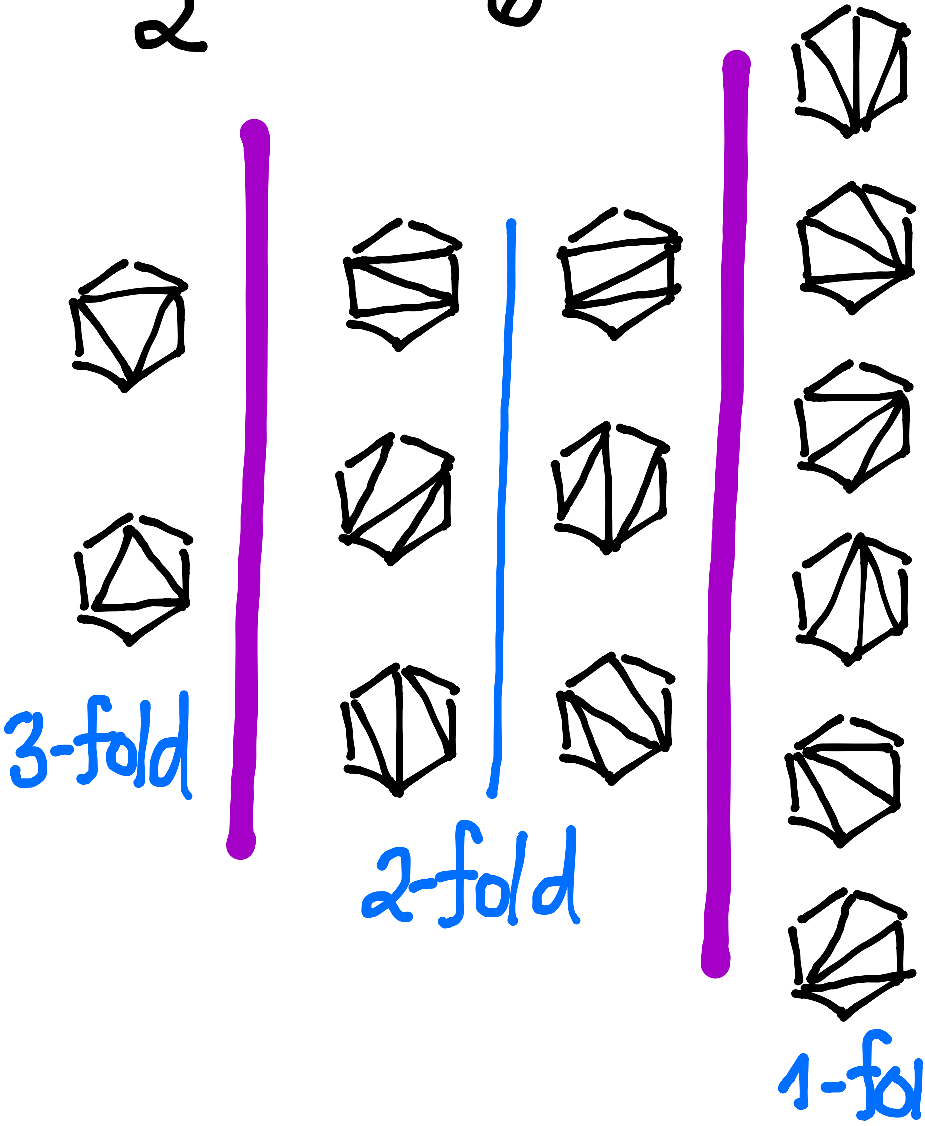
$$1+q^2+q^3+2q^4+q^5+2q^6+q^7+2q^8+q^9+q^{10}+q^{12}$$

$q=e^{\frac{2\pi i}{6}}$ 0

$q=e^{\frac{2\pi i}{3}}$ 2

$q=-1$ 6

$q=1$ 14



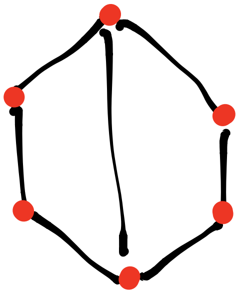
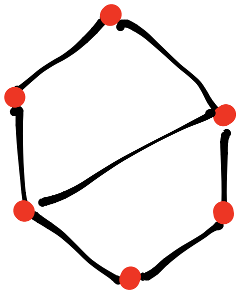
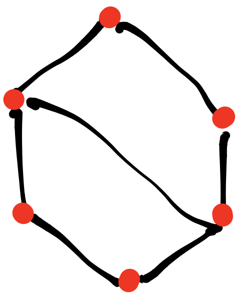
More generally, there are **Fuss-Catalan** numbers

$$C_n^{(m)} = \frac{1}{mn+1}\binom{(m+1)n}{n}$$

counting dissections of an $(mn+2)$-gon into $(m+2)$-gons

e.g. $m=2, n=2$

$$C_2^{(2)} = \frac{1}{5}\binom{3\cdot 2}{2} = 3$$

DEF'N: For $W \leq GL_n(\mathbb{R})$ a finite reflection group with $S^W = \mathbb{R}[f_1, \ldots, f_n]$ and degrees $d_1 \leq \ldots \leq d_n =: h$

- the W-Fuss Catalan number is

$$\mathrm{Cat}^{(m)}(W) := \prod_{i=1}^{n} \frac{mh + d_i}{d_i}$$

- the q-W-Fuss Catalan number is

$$\mathrm{Cat}^{(m)}(W, q) := \prod_{i=1}^{n} \frac{[mh + d_i]_q}{[d_i]_q}$$

THM: (Berest-Etingof-Ginzburg
Gordon 2003)

$\mathrm{Cat}^{(m)}(W)$ lies in $\mathbb{N}$, and $\mathrm{Cat}^{(m)}(W,q)$ lies in $\mathbb{N}[q]$. In fact,

$$\mathrm{Cat}(W,q)=\mathrm{Hilb}\left(\left(S/(\theta_1,\ldots,\theta_n)\right)^W, q\right)$$

where $\theta_1,\ldots,\theta_n$ are a

- homogeneous system of parameters of degree $mh+1$ in S,
- have $\mathbb{R}\theta_1+\ldots+\mathbb{R}\theta_n$ W-stable,
- with same W-repn as $\mathbb{R}x_1+\ldots+\mathbb{R}x_n$.

Why should such magical parameters $\Theta_1, \dots, \Theta_n$ exist??

- In general, need subtle theory of rational Cherednik algebras

Verma $M_{m+\frac{1}{h}}(\mathrm{triv}) \cong S$

simple $L_{m+\frac{1}{h}}(\mathrm{triv}) \cong S/(\Theta_1, \dots, \Theta_n)$

- Even for $W = \mathfrak{S}_n$, it is a bit tricky (Haiman 1993, Dunkl 1998)

- For $W = W(B_n), W(D_n)$ it is easy:

let $(\Theta_1, \dots, \Theta_n) = (x_1^{mh+1}, \dots, x_n^{mh+1})$

Not to be outdone . . .

OBSERVATION:

For $W = GL_n(\mathbb{F}_q) \circlearrowright S = \mathbb{F}_q[x_1, \dots, x_n]$

$$(\Theta_1, \dots, \Theta_n) = (x_1^{q^m}, \dots, x_n^{q^m})$$

- form a homogeneous system of parameters in S, of degree $q^m = [m]_q (q-1) + 1$
- have $\underbrace{\mathbb{F}_q \Theta_1 + \dots + \mathbb{F}_q \Theta_n = \{(c_1 x_1 + \dots + c_n x_n)^{q^m} : \underline{c} \in \mathbb{F}_q^n\}}_{W\text{-stable}}$
- with same W-repn as $\mathbb{F}_q x_1 + \dots + \mathbb{F}_q x_n$!

Clearly then we should consider

$$\mathrm{Hilb}\left(\left(S/\left(x_1^{q^m}, \ldots, x_n^{q^m}\right) \right)^{GL_n(\mathbb{F}_q)}, t \right)$$

as some analogue of $\mathrm{Cat}^{(m)}(W,q)$.

CONJECTURE (Lewis-R.-Stanton 2014)

The above Hilbert series equals

$$\sum_{k=0}^{\min(n,m)} t^{(n-k)(q^m-q^k)} \begin{bmatrix} m \\ k \end{bmatrix}_{q,t}$$

- Proven for $\begin{cases} n = \underbrace{0,1}_{\text{trivial}}, \underset{\substack{\uparrow \\ \text{takes real work!}}}{2} \\ m = \underbrace{0,1}_{\text{easy}}, \underset{\substack{\uparrow \\ \text{recent work of P. Goyal}}}{2} \end{cases}$

- It would imply...

CONJECTURE: The divided power algebra $S^* = \mathrm{Div}(\mathbb{F}_q^n)$ has

$$\mathrm{Hilb}\left(\mathrm{Div}(\mathbb{F}_{q^n})^{GL_n(\mathbb{F}_q)}, t\right) =$$

$$1 + \frac{t^{n(q-1)}}{1-t^{q-1}} + \frac{t^{n(q^2-1)}}{(1-t^{q^2-1})(1-t^{q^2-q})}$$

$$+ \cdots + \frac{t^{n(q^n-1)}}{(1-t^{q^n-1})(1-t^{q^n-q})\cdots(1-t^{q^n-q^{n-1}})}$$

③ Reflection factorizations

THM: (Hurwitz 1891) In $W=\mathfrak{S}_n$, there are

n^{n-2} shortest factorizations

of an n-cycle into transpositions t_i

$$c=(1,2,..,n)=t_1t_2\cdots t_{n-1}$$

THM: (Lewis-R-Stanton 2014) In $W=GL_n(\mathbb{F}_q)$, there are

$(q^n-1)^{n-1}$ shortest factorizations

of a Singer cycle into reflections t_i

$$c=t_1t_2\cdots t_n$$

The proofs can be done in parallel via a method of Frobenius (1896):

In G any finite group, given $\mathcal{C}_1,\dots,\mathcal{C}_\ell \subseteq G$ closed under conjugation,

$$\#\{\text{factorizations } c = c_1 c_2 \cdots c_\ell \text{ with } c_j \in \mathcal{C}_j\}$$

$$= \frac{1}{\#G} \sum_{\substack{\text{irreducible} \\ G\text{-characters } \chi}} \frac{\chi(c^{-1})\,\chi(\mathcal{C}_1)\cdots\chi(\mathcal{C}_\ell)}{\chi(e)^{\ell-1}}$$

where $\chi(\mathcal{C}) := \sum_{g\in\mathcal{C}} \chi(g)$

What makes a reflection factorization $w = t_1 t_2 \cdots t_\ell$ in $GL(V)$ shortest?

Since t_i will fix a hyperplane H_i, w will fix the space $H_1 \cap \dots \cap H_\ell$ of dimension $\geq n-\ell$

Hence $V^w \supseteq H_1 \cap \dots \cap H_\ell$

$$\dim(V^w) \geq n-\ell$$

$$\operatorname{codim}(V^w) \leq \ell$$

THM: In a finite reflection group $W \leq GL_n(\mathbb{R})$
(Carter 1972)

$$w = t_1 t_2 \cdots t_\ell \text{ is shortest}$$
$$\overset{(*)}{\iff} \operatorname{codim}(V^w) = \ell$$

Generally false for complex reflection groups

THM: A finite reflection group $W \leq GL_n(\mathbb{C})$
(Foster-Greenwood 2014)

has $(*)$ $\iff$ either $W \leq GL_n(\mathbb{R})$
or $W = G(d,1,n)$
$= \mathfrak{S}_n[\mathbb{Z}/d\mathbb{Z}]$

THM:
(Huang-Lewis-R.) 2015

General linear groups
$W = GL_n(\mathbb{F})$ for any field $\mathbb{F}$
always have $(*)$.

Carter (1972) actually showed this:

THM: In a finite reflection group $W \leq GL_n(\mathbb{R})$ a reflection factorization $w = t_1 t_2 \cdots t_\ell$ is shortest

$\iff$ (a) the hyperplanes $\underset{V^{t_1}}{\underset{\shortparallel}{H_1}}, \ldots, \underset{V^{t_\ell}}{\underset{\shortparallel}{H_\ell}}$ have

$\dim H_1 \cap \cdots \cap H_\ell = n - \ell$

$\iff$ (b) the lines $\underset{\operatorname{im}(t_1 - 1)}{\underset{\shortparallel}{L_1}}, \ldots, \underset{\operatorname{im}(t_\ell - 1)}{\underset{\shortparallel}{L_\ell}}$ have

$\dim L_1 + \cdots + L_\ell = \ell$

THM: (del Mas 2016) In $W = GL_n(\mathbb{F})$,
a reflection factorization
$w = t_1 t_2 \cdots t_\ell$ is shortest $\iff$

both

(a) the hyperplanes

$$\underset{\substack{\| \\ V^{t_1}}}{H_1}, \ldots, \underset{\substack{\| \\ V^{t_\ell}}}{H_\ell} \text{ have}$$

$$\dim H_1 \cap \cdots \cap H_\ell = n - \ell$$

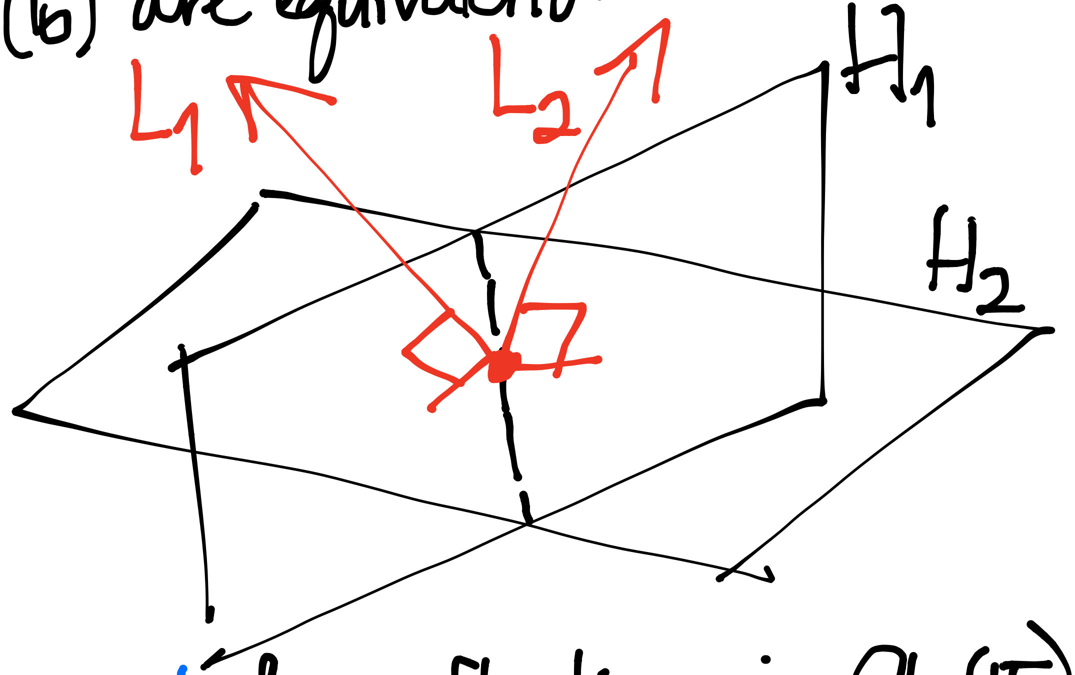
— AND —

(b) the lines

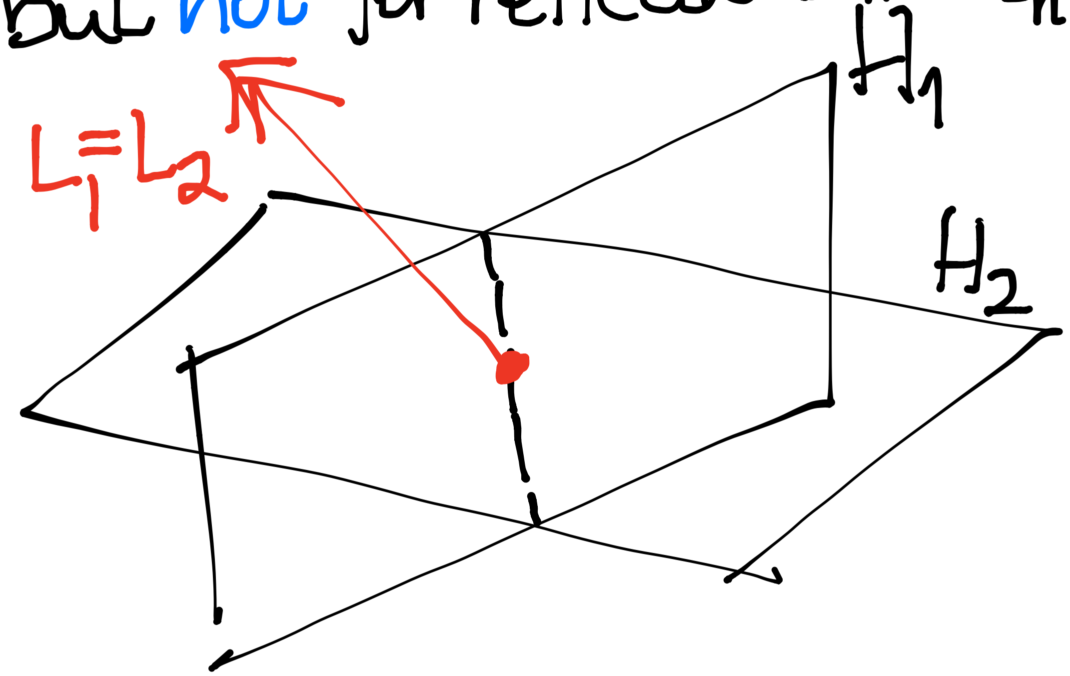
$$\underset{\substack{\| \\ \operatorname{im}(t_1 - 1)}}{L_1}, \ldots, \underset{\substack{\| \\ \operatorname{im}(t_\ell - 1)}}{L_\ell} \text{ have}$$

$$\dim L_1 + \cdots + L_\ell = \ell$$

For orthogonal or unitary reflections, (a), (b) are equivalent:



but not for reflections in $GL_n(\mathbb{F})$:



Thanks to

- Herstmonceux
- Heilbronn Institute
- Tutte Institute
- Fields Institute
- and to you, for your attention!