Orbital diameters of the symmetric and alternating groups

Atiqa Sheikh

Imperial College London

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Introduction

Let $G \leq Sym(\Omega)$ be a finite primitive permutation group.
Consider the natural action of G on $\Omega \times \Omega$ (given by $(\alpha, \beta)^g = (\alpha^g, \beta^g)$).

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Example

The **diagonal** orbital

$$\Delta_0 := \{(\alpha, \alpha) : \alpha \in \Omega\}.$$

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An **orbital graph** of G is an undirected graph Γ_Δ:

$$V(\Gamma) = \Omega$$
$$E(\Gamma) = \{\{\alpha, \beta\} : (\alpha, \beta) \in \Delta\},$$

where Δ is a non-diagonal orbital of G.

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If Δ is an orbital, then we define the **paired** orbital

$$\Delta^* := \{(\alpha, \beta) : (\beta, \alpha) \in \Delta\}.$$

Δ is called **self-paired** if $\Delta = \Delta^*$.

Introduction

Theorem

The following are equivalent:

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Definition

The **orbital diameter** of G is defined to be the maximum of the diameters of the orbital graphs of G. It is denoted by $diam_O(G, \Omega)$.

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Let $\mathcal{C}$ be an infinite class of finite primitive permutation groups. The class $\mathcal{C}$ is said to be **bounded** if there exists an integer $d \geq 1$ such that all the groups in $\mathcal{C}$ have orbital diameters bounded by d.

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Example

1. $G = S_n$, $\Omega = I^{\{k\}}$, the set of k-subsets of $I := \{1, \ldots, n\}$,
   - Orbitals: $\Delta_i = \{(A, B) : |A \cap B| = i\}$, $0 \leq i \leq k$,
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2. $G = S_n$, $\Omega = I^{(k,l)}$, the set of (k, l)-partitions of I,
   - Later: $diam_O(G, \Omega) > \frac{kl}{4} - 1$ NOT BOUNDED class.

Motivation

Question: What are the structures of the primitive groups in such a bounded class $\mathcal{C}$?

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Fact: All finite primitive permutation groups are of one of the following types (due to Aschbacher, O'Nan and Scott):

1. Affine
2. Almost simple
3. Simple diagonal
4. Product action
5. Twisted wreath action

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Theorem (Liebeck, Machpherson, Tent, 2010)

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- $G = S_n$ *or* A_n *acting on* k*-subsets of* $\{1, \ldots, n\}$, k *bounded.*
- $G = Cl_n(q)$ *acting on* k*-subspaces of* $V_n(q)$, k *bounded.*

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Conversely, all such classes of primitive permutation groups are bounded.

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Questions:

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Questions:

1. Given an integer $d \geq 1$, can we find all primitive permutation groups whose orbital diameters are bounded by d?
2. For small d, e.g $d = 2$, can we find all primitive permutation groups for which there exists an orbital graph of diameter d?

Main Results

Fact: All finite primitive permutation groups are of one of the following types (due to Aschbacher, O'Nan and Scott):

1. Affine
2. Almost simple $\rightarrow Soc(G) = A_n \Rightarrow G = A_n$ or S_n $(n \geq 4, n \neq 6)$.
3. Simple diagonal
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Orbital graphs Γ_i, $V(\Gamma_i) = I^{\{k\}}$, $E(\Gamma_i) = \{\{A, B\} : |A \cap B| = i\}$

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Orbital graphs Γ_i, $V(\Gamma_i) = I^{\{k\}}$, $E(\Gamma_i) = \{\{A, B\} : |A \cap B| = i\}$

- $diam_O(G, \Omega) \leq k$
- Orbital graphs: Γ_i - what is $diam(\Gamma_i)$?

Main results: $G = S_n$, action of G on $I^{\{k\}}$

n	$diam(\Gamma_0)$	$diam(\Gamma_1)$	$diam(\Gamma_2)$	$\dots$	$diam(\Gamma_{\lfloor k/2 \rfloor})$	$diam(\Gamma_i)$	$\dots$
$2k+1$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	2	$\left\lceil \frac{k}{k-i} \right\rceil$	$\dots$
$2k+2$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	2	$\vdots$	$\vdots$
$\vdots$	$\left\lceil \frac{n-k-1}{n-2k} \right\rceil$	$\left\lceil \frac{n-k}{n-2k+2} \right\rceil$	$\left\lceil \frac{n-k+1}{n-2k+4} \right\rceil$	$\left\lceil \frac{n-k+(i-1)}{n-2k+2i} \right\rceil$	2	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	2	2	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	2	2	$\vdots$	$\vdots$
$3k-4$	$\vdots$	$\vdots$	2	2	2	$\vdots$	$\vdots$
$3k-3$	$\vdots$	$\vdots$	2	2	2	$\vdots$	$\vdots$
$3k-2$	$\vdots$	2	2	2	2	$\vdots$	$\vdots$
$3k-1$	2	2	2	2	2	$\vdots$	$\vdots$
$3k$	2	2	2	2	2	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Case 2: Action of G on $I^{(k,l)}$, $n = kl$

Let A and B be two (k,l)-partitions of $\{1,\ldots,n\}$. Write $A = A_1|\ldots|A_l$ and $B = B_1|\ldots|B_l$, where $|A_i| = |B_j| = k$. Define the $l \times l$ matrix I_{AB}, by

$$(I_{AB})_{ij} = |A_i \cap B_j|.$$

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If M and N are two $l \times l$ matrices with row and column sums equal to k then we write $N \sim M$ if M can be obtained from N by a permutation of rows and columns. Note that $\sim$ is an equivalence relation. Let $[M]$ denote the equivalence class of M under $\sim$.

Main results: $G = S_n$, action of G on $I^{(k,l)}$, $n = kl$

Orbitals: $\Delta_{[M]} = \{(A, B) : I_{AB} \sim M\}$, where M is an $l \times l$ matrix with row and column sums equal to k.

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Orbital graphs: $\Gamma_{[M]}$, where

$$V(\Gamma_{[M]}) = I^{(k,l)}$$
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Note: Matrices of the form M are called k-doubly stochastic.

Main results: $G = S_n$, action of G on $I^{(k,l)}$, $n = kl$

Example

$l = k = 3$

$$M_1 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, M_2 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}, M_3 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

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$M_4 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Let's look at $\Gamma_{[M]}$, where $M = M_2$.

$$\begin{array}{ccc} A = 123|456|789 & — & B = 143|256|789 \\ | & & | \\ C_1 = 123|567|489 & & C_2 = 147|256|389 \\ | & & | \\ B_1 = 189|234|567 & & B_2 = 147|258|369 \end{array}$$

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So $d(A, B_1) = 2$ and $d(A, B_2) = 3$ and hence $diam(\Gamma_{[M]}) = 3$.

Main results: $G = S_n$, action of G on $I^{(k,l)}$, $n = kl$

Theorem (A.S, 2015)

Let (G, Ω) be a finite primitive permutation group where $G = S_n$ or A_n, and $\Omega = I^{(k,l)}$. Then

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1. $diam_O(G, \Omega) \geq \frac{kl}{4} - 1$. *Hence if* $diam_O(G, \Omega) \leq c$ *then* $n \leq 2c(2c + 1)$.

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2. $diam_O(G,\Omega) \leq 5$ *if and only if* (k,l) *are in the following table:*

l	2	3	4	5	6
k	≤ 11	≤ 5	≤ 3	2	2

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3. *If* $n = k^2$, *let* $\Gamma_{[M]}$ *be the orbital graph where* M *is the* $k \times k$ *matrix with all entries equal to* 1. *Then* $diam(\Gamma_{[M]}) = 2$.

Main results: $G = S_n$, action of G on $I^{(k,l)}$, $n = kl$

Let $M = M_{kl} = \left(\begin{array}{c|cc} kI_{l-2} & 0 & 0 \\ \hline 0 & 1 & k-1 \\ 0 & k-1 & 1 \end{array}\right)$

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Idea of proof.

Suppose k is even and $l = 3$. Write $X = \underbrace{A_1 A_2}_{X_1} | \underbrace{B_1 B_2}_{X_2} | \underbrace{C_1 C_2}_{X_3}$ and

$Y = \underbrace{A_1 B_2}_{Y_1} | \underbrace{B_1 C_2}_{Y_2} | \underbrace{C_1 A_2}_{Y_3}$, where $|A_i| = |B_i| = |C_i| = k/2$.

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Main results: $G = S_n$, action of G on $I^{(k,l)}$, $n = kl$

Proposition (A.S, 2015)

Suppose $k = l$, and let M be the $l \times l$ matrix with all entries equal to 1. Then $diam(\Gamma_{[M]}) = 2$.

Example

$l = k = 4$. Consider $\Gamma_{[M]}$, where $M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$.

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$A = 1\,2\,3\,4|5\,6\,7\,8|9\,10\,11\,12|13\,14\,15\,16$ $\quad$ $B = 1\,2\,15\,16|3\,4\,5\,6|7\,8\,9\,10|11\,12\,13\,14$

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$C = 1\,5\,9\,13|2\,6\,10\,14|3\,7\,11\,15|4\,8\,12\,16$

$I_{AB} = \begin{pmatrix} 2 & 2 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 \\ 2 & 0 & 0 & 2 \end{pmatrix}$

Main results: $G = S_n$, action of G on $I^{(k,l)}$, $n = kl$

Proposition (A.S, 2015)

Suppose $k = l$, and let M be the $l \times l$ matrix with all entries equal to 1. Then $diam(\Gamma_{[M]}) = 2$.

Example

$l = k = 4$. Consider $\Gamma_{[M]}$, where $M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$.

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$$I_{AB} = \begin{pmatrix} 2 & 2 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 \\ 2 & 0 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 1 \end{pmatrix}$$

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Proof uses the fact: every doubly-stochastic matrix has a positive diagonal.

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$$\begin{aligned} |G : H| &\leq 1 + k + k(k-1) + \cdots + k(k-1)^{d-1} \\ &< 1 + k + k^2 + \cdots + k^d < (2|H|)^{d+1}. \end{aligned}$$

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e.g. $d = 2 \Rightarrow n \leq 20 \rightarrow$ *Magma*

Further work

Prove similar results for other almost simple groups whose socle is a simple group of Lie type, e.g. $PSL_n(q)$, $PSp_{2m}(q)$ etc.

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- Almost complete results for $Soc(G) = PSL_2(q)$. Maximal subgroups (in $PSL_2(q)$, q odd) to consider:
  1. $(C_p)^e \rtimes C_{\frac{q-1}{2}}$,
  2. D_{q-1},
  3. D_{q+1},
  4. $PGL_2(q_0)$, where $q = q_0^2$,
  5. $PSL_2(q_0)$, where $q = q_0^r$ and r is an odd prime,
  6. A_4,
  7. A_5,
  8. S_4.

Further work

Theorem (A.S, 2016)

Let $G \leq Sym(\Omega)$ be a finite primitive permutation group with $soc(G) = PSL_2(q)$, $\Omega = (G : H)$ and $\Gamma \leq Out(PSL_2(q))$.

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- *If $diam_O(G, \Omega) \leq 2$ then G must be in the following table:*

G/Γ	H/Γ	*Notes*
$PSL_2({q_0}^2)$	$PGL_2(q_0)$	$q_0 \equiv 3 \pmod 4$
$PSL_2(q)$	$D_{2(q-1)}$	$q = 2^e$, e *odd*, $\Gamma \neq 1$
$PSL_2(q)$	$D_{2(q+1)}$	q *even*
$PSL_2({q_0}^3)$	$PSL_2(q_0)$	q_0 *odd* $\Gamma \neq 1$
$PSL_2({q_0}^r)$	$PSL_2(q_0)$	$q_0 \neq 2$ *even*, $r = 2, 3$

$+$ *some small examples.*

Theorem (continued...)

- *Groups for which there exists families of orbital graphs of diameter 2 include:*

G/Γ	H/Γ	*Notes*
$PSL_2(q)$	D_{q-1}	q *odd*
$PSL_2(q)$	$D_{2(q-1)}$	q *even* > 4
$PSL_2({q_0}^2)$	$PGL_2(q_0)$	$q_0 \equiv 3 \pmod 4$

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Recall: The orbital diameter of G is the maximum of the diameters of all orbital graphs of G.

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Questions:

1. Given an integer $d \geq 1$, can we find all primitive permutation groups whose orbital diameters are bounded by d?
2. For small d, e.g $d = 2$, can we find all primitive permutation groups for which there exists an orbital graph of diameter d?

Thank you!