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Caps and Colouring Steiner Triple Systems

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Abstract. Hill [6] showed that the largest cap in $\mathbb{P}G(5,3)$ has cardinality 56. Using this cap it is easy to construct a cap of cardinality 45 in $\mathbb{A}G(5,3)$. Here we show that the size of a cap in $\mathbb{A}G(5,3)$ is bounded above by 48. We also give an example of three disjoint 45-caps in $\mathbb{A}G(5,3)$. Using these two results we are able to prove that the Steiner triple system $\mathbb{A}G(5,3)$ is 6-chromatic, and so we exhibit the first specific example of a 6-chromatic Steiner triple system.

Keywords: caps, Steiner triple systems, colouring

Let $\mathbb{F}_3^n$ denote the vector space of dimension n over $\mathbb{F}_3$, the field of order 3, and let $\mathbb{A}G(n,3)$ be the set of all cosets of $\mathbb{F}_3^n$. Then $\mathbb{A}G(n,3)$ is called the *affine geometry of dimension n over* $\mathbb{F}_3$. For $k = 0, \ldots, n$, a k-flat of $\mathbb{A}G(n,3)$ is a coset of a subspace of dimension k. The points of $\mathbb{A}G(n,3)$ are the 0-flats and they are identified with the vectors of $\mathbb{F}_3^n$. The projective geometry $\mathbb{P}G(n,3)$ is defined as the space of equivalance classes $(\mathbb{A}G(n+1,3)\backslash\{\vec{0}\})/\sim$ where $x \sim y$ if $\exists c \in \mathbb{F}_3$ such that $x = cy$. For $k \geq 0$, the image of a $(k+1)$-flat in $\mathbb{A}G(n,3)$ is defined to be a k-flat of $\mathbb{P}G(n,3)$. In both $\mathbb{A}G(n,3)$ and $\mathbb{P}G(n,3)$, the 1-flats are called *lines*, the 2-flats are called *planes* and the $(n-1)$-flats are called *hyperplanes*.

A subset of $\mathbb{A}G(n,3)$ or $\mathbb{P}G(n,3)$ is called a *cap* if no three of its points are collinear, i.e., if no three of its points lie in the same 1-flat. A cap of cardinality k is called a k-cap. We will need some results about the size and structure of caps in $\mathbb{A}G(4,3)$ and $\mathbb{A}G(5,3)$. We denote by $\beta(\mathbb{A}G(n,3))$ the largest integer k for which there exists a k-cap in $\mathbb{A}G(n,3)$. It is easy to see that $\beta(\mathbb{A}G(1,3)) = 2$ and $\beta(\mathbb{A}G(2,3)) = 4$. That $\beta(\mathbb{A}G(3,3)) = 9$ is well known (see [8]). Pellegrino [9] showed that $\beta(\mathbb{A}G(4,3)) = 20$. In this paper we show that $45 \leq \beta(\mathbb{A}G(5,3)) \leq 48$.

A Steiner triple system of order v (an STS(v)) is a pair $S = (V(S), T)$ where $V(S)$ is a v-set and T is a collection of triples (subsets of $V(S)$ of cardinality 3) such that each pair of distinct elements of S lies in exactly one triple of T. It is well known that a STS(v)

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exists iff $v \equiv 1$ or 3 (mod 6), such an integer is called *admissible*. Moreover it is easy to verify that the set of all points of $\mathbb{A}G(n,3)$ together with its lines forms an STS(3^n). An r-colouring of a STS, $S = (V(S), T)$, is a partitioning of the vertex set $V(S)$ into r disjoint subsets such that no triple of T is contained entirely in one of these r subsets. If an STS(v), S, has an r-colouring but no $(r-1)$-colouring then we say that S is r-chromatic. We point out in passing that the existence of STS's with prescribed chromatic number $r \geq 5$ is far from being straightforward and is shown (only) by non-constructive methods in [1], and moreover, the first specific example of a 5-chromatic STS is given in [3]. It is often extremely hard to show that a given STS cannot be k-coloured when $k \geq 4$ (cf. the calculation done in [3]). Here we will give a relatively short proof that $\mathbb{A}G(5,3)$ does not admit a 5-colouring.

Let $\phi(n)$ denote the number of hyperplanes in $\mathbb{A}G(n,3)$, and for $j = 1, 2$, let $\phi(j,n)$ denote the number of distinct hyperplanes passing through j fixed points. It is well known (e.g., see [10]) that

$$\phi(2,n) = 3^{n-3}\frac{3^n-1}{3^{n-2}-3^{n-3}}\ ,\ \ \phi(1,n) = 3^{n-2}\frac{3^n-1}{3^{n-1}-3^{n-2}}\ ,\ \ \phi(n) = 3\phi(1,n).$$

Now if 3 points do not form a line, then they determine a unique plane and there are $\prod_{2\leq \ell \leq n-2}\frac{(3^n-3^\ell)}{(3^{n-1}-3^\ell)}$ distinct hyperplanes through any plane in $\mathbb{A}G(n,3)$ where $n \geq 4$ (this number is clearly one if $n = 3$). Let $n \geq 4$ and $C \subseteq \mathbb{A}G(n,3)$ be a set of cardinality t. For $0 \leq i \leq t$, define $n_i := |\{K$ such that K is a hyperplane and $|K \cap C| = i\}|$. Consider the following equations (see [4] and [7]): Clearly

$$\sum_i n_i = 3^{n-1}\frac{3^n-1}{3^{n-1}-3^{n-2}}$$

and counting the number of pairs $\{p, K\}$ where p is a point of the hyperplane K and $p \in C$ we find

$$\sum_i i n_i = 3^{n-2}\frac{3^n-1}{3^{n-1}-3^{n-2}}t.$$

Now counting triples $\{p_1, p_2, K\}$ such that K is a hyperplane and $\{p_1, p_2\} \subseteq C \cap K$ gives

$$\sum_i \binom{i}{2} n_i = 3^{n-3}\frac{3^n-1}{3^{n-2}-3^{n-3}}\binom{t}{2}.$$

Moreover suppose that the set C is a cap, then by counting quadruples $\{p_1, p_2, p_3, K\}$ where K is a hyperplane and $\{p_1, p_2, p_3\} \subseteq C \cap K$ we obtain

$$\sum_i \binom{i}{3} n_i = \prod_{2\leq \ell \leq n-2}\frac{(3^n-3^\ell)}{(3^{n-1}-3^\ell)}\binom{t}{3}.$$

Consider now a cubic polynomial $P(i) := (i-r_1)(i-r_2)(i-r_3)$. Then there are numbers $a_3, \ldots, a_0$ such that $P(i) = a_3\binom{i}{3} + a_2\binom{i}{2} + a_1 i + a_0$. Thus

$$\sum_i P(i) n_i \ = \ a_3 \sum_i \binom{i}{3} n_i + a_2 \sum_i \binom{i}{2} n_i + a_1 \sum_i i n_i + a_0 \sum_i n_i$$

$$= \; a_3 \prod_{2\le \ell \le n-2} \frac{(3^n - 3^\ell)}{(3^{n-1} - 3^\ell)} \binom{t}{3} + a_2 3^{n-3} \frac{3^n - 1}{3^{n-2} - 3^{n-3}} \binom{t}{2}$$
$$+ a_1 3^{n-2} \frac{3^n - 1}{3^{n-1} - 3^{n-2}} t + a_0 3^{n-1} \frac{3^n - 1}{3^{n-1} - 3^{n-2}} \; .$$

This sum is a cubic polynomial of t, which is denoted $f_P(t)$. We may use $f_P(t)$ to study C. For example we may choose the roots r_1, r_2 and r_3 of $P(i)$ such that if $n_i \neq 0$ then $P(i) \geq 0$. With such a choice, $f_P(t)$ is necessarily nonnegative and this gives information about the value of t.

Now Pellegrino showed in [9] that the largest caps in $\mathbb{P}G(4, 3)$ have cardinality 20. Since one of the caps he constructed actually lies in $\mathbb{A}G(4, 3) \subset \mathbb{P}G(4, 3)$, this proved $\beta(\mathbb{A}G(4, 3)) = 20$. In [5] Hill classified all nonisomorphic 20-caps in $\mathbb{P}G(4, 3)$. Only one of the inequivalent 20-caps is contained in an affine subspace of $\mathbb{P}G(4, 3)$, and its intersections with the 120 different hyperplanes of $\mathbb{A}G(4, 3)$ give the following values for the n_i

$$n_2 = 10, \quad n_6 = 60, \quad n_8 = 30, \quad n_9 = 20 \quad \text{and} \quad n_i = 0 \text{ for } n \neq 2, 6, 8, 9. \qquad (1)$$

A direct proof of these facts can be found in [7]. It follows easily from (1) that if C is a 20-cap in $\mathbb{A}G(4, 3)$, and L_1, L_2 and L_3 are three parallel hyperplanes, then $\{|C \cap L_1|, |C \cap L_2|, |C \cap L_3|\} \in \{\{2, 9, 9\}, \{6, 6, 8\}\}$. Moreover, in [6], Hill shows that the largest cap in $\mathbb{P}G(5, 3)$ has cardinality 56 and this cap is unique up to isomorphism. This fact is reported in the conclusion of [2] where it is conjectured that $\beta(\mathbb{A}G(5, 3)) = 45$. We will show that $45 \leq \beta(\mathbb{A}G(5, 3)) \leq 48$. Hill showed that if C is his 56-cap and H is any hyperplane of $\mathbb{P}G(5, 3)$ then $|C \cap H| \in \{11, 20\}$. If we choose H with $|C \cap H| = 11$ then $C \backslash (C \cap H)$ is a 45-cap in $(\mathbb{P}G(5, 3)\backslash H) \cong \mathbb{A}G(5, 3)$.

To prove that $\beta(\mathbb{A}G(5, 3)) \leq 48$ we first need the following lemma.

LEMMA 1 *Let C be a 49-cap in $\mathbb{A}G(5, 3)$. Then there is a hyperplane H of $\mathbb{A}G(5, 3)$ such that $|C \cap H| = 9$.*

Proof. First note that there is no hyperplane H_0 such that $|C \cap H_0| \leq 8$, as otherwise one of the two hyperplanes parallel to H_0 would have to contain at least 21 elements of C. Consider now the polynomial $P(i) := (i - 11)(i - 17)(i - 18)$. Note that $P(i) \geq 0$ for all integers $i \geq 11$. Thus $n_i P_i \geq 0$ for all integers i except possibly $i = 9, 10$. Now

$$\sum_i P(i) n_i = f_P(49) = -92. \qquad (2)$$

Hence at least one of n_9 or n_{10} is nonzero. Now if H_1 is any hyperplane such that $|C \cap H_1| = 10$, then the two hyperplanes H_2 and H_3 parallel to it satisfy $\{|C \cap H_2|, |C \cap H_3|\} = \{19, 20\}$. Thus $n_{19} \geq n_{10}$ and $n_{20} \geq n_{10}$. Suppose now that $n_9 = 0$. Then from (2) we have $P(10)n_{10} + P(19)n_{19} + P(20)n_{20} + A = -92$, where $A \geq 0$. As $n_{19} \geq n_{10}$ and $n_{20} \geq n_{10}$, $P(10) = -56$, $P(19) = 16$ and $P(20) = 54$, we deduce that $-92 \geq P(10)n_{10} + P(19)n_{19} + P(20)n_{20} \geq P(10)n_{10} + P(19)n_{10} + P(20)n_{10} = (-56 + 54 + 16)n_{10}$, a contradiction. Hence $n_9 \neq 0$. ∎

LEMMA 2 $\beta(\mathbb{A}G(5,3)) \leq 48$

Proof. Let C be a cap in $\mathbb{A}G(5,3)$ of size 49. By the lemma above, there is a hyperplane H_1 such that $|C \cap H_1| = 9$. Now the two hyperplanes H_2 and H_3 parallel to H_1 satisfy $|C \cap H_2| = |C \cap H_3| = 20$, and thus $C \cap H_i$ is a maximal cap of H_i for $i = 2, 3$. Take K_1 another hyperplane with K_2 and K_3 the two hyperplanes parallel to K_1, and define $L_{ij} := H_i \cap K_j \cong \mathbb{A}G(3,3)$ for $1 \leq i, j \leq 3$. In addition to $H_i = L_{i1} \cup L_{i2} \cup L_{i3}$ and $K_j = L_{1j} \cup L_{2j} \cup L_{3j}$ we will also consider the 6 hyperplanes $L(i, j, k) := L_{1i} \cup L_{2j} \cup L_{3k}$ where $\{i, j, k\} = \{1, 2, 3\}$.

Choose K_1 so that $|C \cap L_{21}| = 2$. From (1), we have that $(|C \cap L_{21}|, |C \cap L_{22}|, |C \cap L_{23}|) = (2, 9, 9)$. Moreover, as mentioned earlier, we have $\{|C \cap L_{31}|, |C \cap L_{32}|, |C \cap L_{33}|\} \in \{\{2, 9, 9\}, \{6, 6, 8\}\}$. Suppose first that $(|C \cap L_{31}|, |C \cap L_{32}|, |C \cap L_{33}|) = (6, 6, 8)$. Then considering $L(1, 2, 3)$ we see that $|C \cap L_{11}| \leq 3$ and by (1) equality cannot hold, thus $|C \cap L_{11}| \leq 2$. Similarly, considering K_3 shows that $|C \cap L_{13}| \leq 2$, and thus $|C \cap L_{12}| \geq 5$. Thus the hyperplane K_2 either contains at least 21 elements of C or meets C in 20 points including exactly 5 points lying in the hyperplane L_{12} of K_2, contradicting (1). The cases $(|C \cap L_{31}|, |C \cap L_{32}|, |C \cap L_{33}|) \in \{(8, 6, 6), (6, 8, 6)\}$ are similar. On the other hand if $(|C \cap L_{31}|, |C \cap L_{32}|, |C \cap L_{33}|) = (2, 9, 9)$, using $L(1, 2, 3)$, L_2 and L_3 we see that $|C \cap L_{1j}| \leq 2$ for $j = 1, 2, 3$ a contradiction with $|C \cap H_1| = 9$. The cases $(|C \cap L_{31}|, |C \cap L_{32}|, |C \cap L_{33}|) \in \{(9, 2, 9), (9, 9, 2)\}$ are similar. ∎

COROLLARY 3 *The STS(243)* $\mathbb{A}G(5,3)$ *cannot be 5-coloured.*

Proof. Assume that $\mathbb{A}G(5,3) = C_1 \sqcup C_2 \sqcup C_3 \sqcup C_4 \sqcup C_5$ is a 5-colouring of $\mathbb{A}G(5,3)$ with $|C_1| \geq |C_i|$ for $i = 2, 3, 4, 5$. Since $|\mathbb{A}G(5,3)| = 243$, we see that C_1 is a cap containing at least $\lceil \frac{243}{5} \rceil = 49$ points. ∎

THEOREM 4 $\mathbb{A}G(5,3)$ *is a 6-chromatic STS(243).*

Proof. The mathematical software MapleV was used to discover the following three pairwise disjoint 45-caps C_1, C_2 and C_3 of $\mathbb{A}G(5,3)$. Then a computer program was used to 3-colour the partial STS obtained by restricting $\mathbb{A}G(5,3)$ to the $108 = 243 - 3(45)$ remaining points. We obtained the following 6-colouring of $\mathbb{A}G(5,3)$.

C_1 := {02201, 02101, 12202, 01211, 21111, 00120, 10221, 00112, 21100, 20210, 11211, 00002, 02020, 10020, 21000, 00010, 11110, 21210, 20120, 11121,00212, 00201, 22220, 02220, 20001, 22001, 21221, 21101, 10112, 22222, 22212, 00110, 02021, 22121, 10111, 21220, 01210, 02102, 20100, 01102, 01110, 22021, 02200, 11221, 22101};

C_2 := {00200, 11112, 10102, 01120, 00222, 20121, 00100, 11101, 02222, 11212, 22012, 20022, 22200, 12220, 22211, 02221, 01202, 10212, 22022, 21122, 22122, 21201, 22210, 02120, 10011, 01201, 00111, 20111, 02011, 21211, 02211, 01101, 00101, 00001, 20201, 21121, 21021, 10100, 00020, 22112, 02012, 21212, 21102, 11202, 02122};

C_3 := {02000, 20010, 22202, 10121, 10002, 12211, 22000, 11122, 21110, 02111, 02202, 21202, 20102, 02212, 11200, 02110, 22100, 01220, 00122, 11220, 01122, 22110, 01222, 21012, 22201, 21120, 00022, 21200, 00102, 11100, 10200, 22010, 00210, 02210, 10120, 00011, 01111, 00121, 00221, 22221, 02002, 20112, 21112, 20222, 21222};

C_4 := {11021, 21002, 01022, 10022, 21022, 12120, 02121, 12212, 11020, 20020, 22120, 00000, 10000, 12002, 01000, 12000, 12100, 01200, 11120, 01112, 11010, 20220, 01002, 10122, 20021, 12022, 02010, 11201, 10201, 10202, 12101, 20110, 00202, 22002, 20212, 12222};

C_5 := {20000, 11000, 02100, 12110, 10210, 11210, 10001, 01001, 21001, 12001, 10101, 12201, 01011, 21011, 12011, 22011, 00211, 10211, 01121, 20221, 12221, 20002, 11002, 11102, 12102, 22102, 20202, 00012, 10012, 01012, 12012, 12112, 01212, 11022, 02022, 20122};

C_6 := {01100, 20200, 12200, 10010, 01010, 21010, 12010, 10110, 12210, 01020, 21020, 12020, 22020, 00220, 10220, 11001, 02001, 20101, 20011, 11011, 11111, 12111, 22111, 20211, 00021, 10021, 01021, 12021, 12121, 01221, 20012, 11012, 02112, 12122, 10222, 11222}. ■

It is shown in [3] that if $v \equiv 3 \pmod 6$ and if there exists an r-chromatic STS(v), then there exists an r-chromatic STS(u) for every admissible $u \geq 2v + 1$. Combining this fact with Theorem 4 we get

COROLLARY 5 *There exists a 6-chromatic STS(u) for every admissible $u \geq 487$.*

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