On Games with Coupled Constraints

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***Abstract*—We study a general cost minimization game in which each player minimizes the cost of its resource consumption while achieving a target utility level. The player strategies are coupled through both their cost functions and their utility functions. Equilibrium exists only for certain target utility levels, and is characterized by the equilibrium of a dual game in which each player maximizes its utility while keeping the cost of its resource consumption below a cost threshold. We show that the dual game possesses equilibrium under very mild conditions, in particular with no a priori assumption on the compactness of player strategy sets. We also obtain an inner estimate of the set of equilibrium utility levels in the case of decoupled cost functions by a minimax approach. We then relax the hard constraint on achieving a target utility level, and introduce an unconstrained weighted cost minimization game which always possesses equilibrium. Under mild conditions, we recover the original equilibrium as the penalty on not achieving the target utility levels increases. Finally, we discuss the possibility of learning to play an equilibrium strategy via the best response dynamics.**

I. Introduction

Since the publication of Nash's famous papers [1], [2] on the theory of noncooperative games, game theoretic methods have been widely used in studying problems of strategic decision making by multiple self-interested players [3]. Although early applications of game theory has primarily been in economics and social sciences, it later found applications in such diverse areas as biology, engineering, and computer science. Loosely speaking, a game involves a group of players where each player chooses a strategy from its own set of strategies to maximize its own utility. In a typical game scenario, each player's utility is determined by the joint strategies chosen by all players; however, the set of strategies available to each player is fixed and independent of the strategies chosen by the other players. Despite their great modeling capabilities, such game-theoretic models are still not suitable for strategic engagements where a player's set of strategies may be constrained by the strategies chosen by the other players.

Shortly after the appearance of Nash's papers, what one might call "a game with coupled constraints" is introduced by Debreu in [4]. One salient aspect of the model introduced in [4] is the dependence of each player's set of strategies on the strategies chosen by the other players. In the follow-up paper [5], Arrow and Debreu coined the term "abstract economy" for the same model and presented a refinement of the equilibrium existence result in [4]. Since then, numerous papers focused on games with coupled constraints to study equilibrium and its properties as well as to develop numerical and iterative schemes leading to equilibrium. We refer the reader to the survey article [6] for a historical development of the various methods to address such problems. Currently, the literature on games with coupled constraints in contrast to games with fixed strategy sets is much smaller, arguably because of the challenges posed by the constraint coupling.

One special case of considerable interest is where the joint player strategies are constrained to a convex subset of the joint strategy space [7], [8]. This special case, known as "jointly convex case" or "shared constraints case", appears to be first treated by Rosen in [7] where, in addition to the existence of equilibrium under general assumptions, the uniqueness of equilibrium along with the convergence of projected gradient dynamics is shown under strong concavity assumptions on the player utilities. Afterwards, many studies have been conducted to advance the state of the art in this special case; see [9] for a relaxation approach based on the Nikaido-Isoda function, [10], [11], [12] for a variational inequality approach, [13], [14], [15] for a penalty approach (where the general case is also considered), and [6] for a general survey. However, many interesting applications including the one presented in this paper do not fall into this special case.

The application motivating this paper is the problem of distributed power minimization in MIMO interference networks introduced in [16] and summarized in Section II. The setup for this application leads to a game with coupled constraints from which we abstract out a general cost minimization game, a one-shot simultaneous move game. In our cost minimization game, each player selects its strategy to achieve its target utility level with minimum cost. The strategies are considered to be different types of resources available to the players at any amount. As a result, the player strategy sets are unbounded and therefore not compact. Furthermore, there is no obvious way of imposing bounds on the strategy sets without changing the equilibrium structure of the cost minimization game.

Most of the current literature on the games with coupled constraints assume compactness of the player strategy sets; see [6] and the references therein. In addition, most of the results in the literature that do not make such a compactness assumption make some form of a coercivity assumption under which the players' optimal response strategies can effectively be restricted to a compact subset of the original set of joint strategies; see for example Theorem 2 in [17]. Finding such a compact subset that is stable under players' optimal response mappings requires an equilibration process which is not an easy task to accomplish in our cost minimization game. Another approach which does not rely on the compactness of strategy sets is via degree theory; see [18], in particular Theorem 12.1. However, satisfying the hypotheses of this

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approach in our cost minimization game does not appear to be within reach. Regarding the exact penalty approach pursued in [13], [15] (without boundedness of the strategy sets), the results in these papers do not the address the issue of existence of equilibrium, and are obtained under various differentiability assumptions on the objective functions and the functions defining the feasible set of strategies. The other work in this area that do not require the compactness of the strategy sets focus on more specific models and exploit the special structure of the problem; see for example [12] which focuses on scaled congestion costs and shared constraints, and [19] which focuses on the problem of power minimization in parallel interference channels.

In sum, the cost minimization games considered in this paper constitute a fairly broad class within the family of games with coupled constraints. Generally, a cost minimization game does not fall into any of the special cases considered in the literature, and it is not readily amenable to the existing results in the literature. Hence, the main contribution of this paper is to advance the state of the art with respect to a large class of games with coupled constraints, called cost minimization games. More precisely, our contributions on cost minimization games include the following.

- A duality relation with a utility maximization game is presented, which allows us to identify the original equilibria with the equilibria of a dual game. It is shown that any dual game possesses equilibrium under very natural assumptions, in particular with no *a priori* assumption on the compactness of the strategy sets. (A duality result for games with shared constraints and bounded strategy sets is reported in [8], however, the duality notion in [8] is a game-theoretic extension of the Lagrangian duality, and is quite different from the duality notion in this paper.)
- By exploiting the duality relation, a minimax approach is presented for the case of decoupled cost functions, which leads to explicit sufficient conditions for the existence of equilibrium. The benefits of the minimax approach is illustrated by an application on power minimization in MIMO interference systems for the special case of diagonal channel matrices. In this application, the minimax approach readily yields a sufficient condition for the existence of equilibrium, which is a relaxation of the main existence condition obtained in [19] using an advanced degree-theoretic result.
- An exact penalty approach is presented by penalizing the constraint violations and thereby removing the constraints. Sufficient conditions under which the original equilibria is recovered through the equilibria of an unconstrained game are presented. (Some of our results in this approach are similar to those presented in [13], [15]; however, our results require less stringent conditions for our cost minimization game, for example, our results do not require differentiability of the cost or the utility functions.)
- The convergence of the best response dynamics to an equilibrium is shown for the case of weakly coupled games. Examples of nonconvergence are also provided. (Similar algorithms, called iterative water-filling algorithms, are shown to be convergent in [19] for the special case of diagonal channel matrices in the problem of power minimization in MIMO interference systems.)

The remainder of this paper is organized as follows. Section II presents a motivating application. Section III introduces a cost minimization game, whereas Section IV introduces a utility maximization game and presents a duality relation. Section V focuses on the case of decoupled cost functions. Section VI is devoted to an exact penalty approach. Section VII discusses the best response dynamics and the issue of convergence to an equilibrium. Section VIII presents some simulation results. Finally, Section IX concludes the paper with some final remarks.

A. Notation

$:=$ stands for "defined as"
$\equiv$ stands for "identically equal to"
$\mathbb{R}$ denotes the set of real numbers
$\mathbb{R}^d$ denotes the $d-$dimensional Euclidean vector space
$\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x_i \geq 0, \text{ for all } i \in \{1, \ldots, d\}\}$
$\mathbb{R}^d_{++} = \{x \in \mathbb{R}^d : x_i > 0, \text{ for all } i \in \{1, \ldots, d\}\}$
$\leq, \geq, <, >$ are interpreted elementwise for real vectors
$-k$ denotes the set of indices other than k;
e.g., $x_{-k} = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_d)$
$(\cdot)^+ = \max\{\cdot, 0\}$
$\langle \cdot, \cdot \rangle$ denotes the standard inner product in a Euclidean space
$\|\cdot\|$ denotes the standard norm in a Euclidean space
$\dagger$ denotes the conjugate transpose
I denotes an identity matrix of proper dimension
$\det(\cdot)$ denotes the determinant for a square matrix
$\mathcal{H}_+$ denotes the set of complex positive semi-definite matrices of proper dimension
$E(.)$ denotes the expectation
$\emptyset$ denotes the empty set
$\mathrm{cc}(\cdot)$ denotes the closure of the convex hull
$\mathcal{R}_{\mathcal{S}}(s) = \{x \in \mathbb{R}^d : \text{ there exists } \epsilon \in \mathbb{R}_{++} \text{ such that } s + \epsilon x \in \mathcal{S}\}$, for convex $\mathcal{S} \subset \mathbb{R}^d$, $s \in \mathcal{S}$
$\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{Y}$ indicates that $\mathcal{F}$ is a correspondence from $\mathcal{X}$ to the set of subsets of $\mathcal{Y}$
$\Rightarrow$ stands for "implies";
e.g., $A \Rightarrow B$ means that A implies B
$\Leftrightarrow$ stands for "is equivalent to"
e.g., $A \Leftrightarrow B$ means that A is equivalent to B

II. A MOTIVATING APPLICATION

We introduce a problem of power optimization in a MIMO interference system, studied in our previous work [16], as a motivating application for this paper. A MIMO interference system consists of P communication links where each link has a transmitter and a receiver; see [19], [20], [21], [22], [23], [24], [25], [26], [27]. Each transmitter and each receiver is equipped with multiple antennas. The user of each link p sends a complex signal vector x_p; as a result, a complex signal vector y_p is received at the receiver of link p. The received signal vector y_p is given as

$$y_p = H_{p,p}x_p + \sum_{q \neq p} H_{p,q}x_q + z_p$$

where

- $H_{p,q}$ is the complex channel matrix between the $q-$th transmitter and the $p-$th receiver,
- z_p is the circularly symmetric complex Gaussian noise vector at the $p-$th receiver with $E(z_p) = 0$ and $E(z_p z_p^\dagger) = I$.

The user of each link p decides on the distribution of x_p to satisfy a Quality of Service (QoS) requirement

$$I(x_p; y_p) \geq \bar{u}_p \tag{1}$$

with minimum power consumption

$$E(x_p^\dagger x_p)$$

where $I(x_p; y_p)$ is the mutual information for the link p and $\bar{u}_p \in \mathbb{R}_+$ is a target rate. The user of each link p views its total interference $\sum_{q \neq p} H_{p,q}x_q$ as a zero-mean circularly symmetric complex Gaussian noise vector. In this case, the power consumption $E(x_p^\dagger x_p)$ is minimized by a zero-mean circularly symmetric complex Gaussian distribution satisfying (1); see [20]. Moreover, the mutual information for link p takes the form

$$I(x_p; y_p) = \log_2 \det\bigl(I + R_p^{-1/2} H_{p,p} Q_p H_{p,p}^\dagger R_p^{-1/2}\bigr)$$

where

$$R_p := I + \sum_{q \neq p} H_{p,q} Q_q H_{p,q}^\dagger$$

and $Q_p := E(x_p x_p^\dagger) \in \mathcal{H}_+$; see [20].

Therefore, the power minimization problem for the user of each link p reduces to choosing a covariance matrix Q_p from the set of feasible strategies

$$\{Q_p \in \mathcal{H}_+ : \bar{u}_p - I(x_p; y_p) \leq 0\} \tag{2}$$

in order to minimize the cost

$$\text{trace}(Q_p).$$

Note that the set of feasible strategies for each link p in (2) depends on the decisions Q_{-p} of the other links even though the cost trace(Q_p) of each link p depends only on its own decision Q_p. This means that the users of P links are engaged in a noncooperative game with coupled constraints which motivates us to introduce a general cost minimization game in the next section.

III. A Cost Minimization Game

Motivated by the power optimization problem introduced in Section II, we abstract out a general cost minimization game. We start with a finite player set $\mathcal{P} := \{1, \ldots, P\}$. Each player $p \in \mathcal{P}$ has a strategy set $\mathcal{S}_p$ which is a nonempty closed convex cone (with vertex at 0) in $\mathbb{R}^{d_p}$ where $d_p \geq 1$ is a finite integer. It is convenient to think of $\mathcal{S}_p$ as the set of resources available to player p. Hence, for any non-zero $s_p \in \mathcal{S}_p$, the semi-infinite ray $\mathbb{R}_+ s_p := \{\hat{s}_p \in \mathcal{S}_p : \hat{s}_p = \alpha s_p,\ \alpha \in \mathbb{R}_+\}$ can be interpreted as the set of resources of the same type as that of s_p but with different intensities. Whereas, if $s_p^1, s_p^2 \in \mathcal{S}_p$ are linearly independent, then s_p^1 and s_p^2 can be interpreted as different types of resources. In our cost minimization game, all players choose their strategies simultaneously and their collective choices are represented by some profile of strategies $s \in \mathcal{S}$ where $\mathcal{S} := \bigtimes_{p \in \mathcal{P}} \mathcal{S}_p$. For any $s \in \mathcal{S}$ and $p \in \mathcal{P}$, $s_p \in \mathcal{S}_p$ denotes the strategy chosen by player p, i.e., $p-$th entry in s, whereas $s_{-p} \in \mathcal{S}_{-p}$ denotes the profile of strategies chosen by all players other than player p, where $\mathcal{S}_{-p} := \bigtimes_{q \in \mathcal{P}-\{p\}} \mathcal{S}_q$. We sometimes write $s \in \mathcal{S}$ as $s = (s_p, s_{-p})$, for some $p \in \mathcal{P}$.

If all players choose the strategy profile $s \in \mathcal{S}$, then each player $p \in \mathcal{P}$ incurs the cost $c_p(s)$ and receives the utility $u_p(s)$ where $c_p : \mathcal{S} \mapsto \mathbb{R}_+$ and $u_p : \mathcal{S} \mapsto \mathbb{R}_+$ denote player p's cost and utility functions, respectively. We make the following assumption throughout the paper without further mention.

Assumption 1: For all $p \in \mathcal{P}$, $(s_p, s_{-p}) \in \mathcal{S}$, and $h_p \in \mathcal{R}_{\mathcal{S}_p}(s_p)$,

(i) c_p, u_p are continuous in $\mathcal{S}$

(ii) $c_p(\cdot, s_{-p})$ is convex in $\mathcal{S}_p$, $u_p(\cdot, s_{-p})$ is concave in $\mathcal{S}_p$

(iii) $c_p(0, s_{-p}) = u_p(0, s_{-p}) = 0$

(iv) if $s_p \neq 0$, then $c_p(\alpha s_p, s_{-p})$, $u_p(\alpha s_p, s_{-p})$ are (strictly) increasing with respect to $\alpha \in \mathbb{R}_+$

(v) there exists some $M \geq 0$, such that

$$\inf_{\hat{s} \in \mathcal{S} : \|\hat{s}_p\| \geq M} c_p(\hat{s}) > 0$$

(vi) $c_p(\cdot, s_{-p})$ and $u_p(\cdot, s_{-p})$ possess one-sided directional derivatives

$$c_p'(s_p, s_{-p}; h_p) := \lim_{\epsilon \downarrow 0} \frac{c_p(s_p + \epsilon h_p, s_{-p}) - c_p(s_p, s_{-p})}{\epsilon}$$

$$u_p'(s_p, s_{-p}; h_p) := \lim_{\epsilon \downarrow 0} \frac{u_p(s_p + \epsilon h_p, s_{-p}) - u_p(s_p, s_{-p})}{\epsilon}$$

where $c_p'(s_p, s_{-p}; h_p) \in \mathbb{R}$ and $u_p'(s_p, s_{-p}; h_p) \in \mathbb{R}$.

Part (i) and part (ii) of Assumption 1 are for technical reasons. Part (iii) means that using no resources has no cost and yields no utility. Part (iv) ensures sensible behavior expected of cost and utility functions in a resource allocation problem. Part (v) requires that the cost of any resource $\hat{s}_p \in \mathcal{S}_p$ with large enough intensity is uniformly higher than a certain nonzero level for all $\hat{s}_{-p} \in \mathcal{S}_{-p}$. Part (vi) is assumed for simplicity and it can be relaxed[1]. We observe that, under Assumption 1, the following are true: for all $p \in \mathcal{P}$, $(s_p, s_{-p}) \in \mathcal{S}$ with $s_p \neq 0$,

- $0 \in \mathcal{S}_p$ is the unique global minimizer of $c_p(\cdot, s_{-p})$ and $u_p(\cdot, s_{-p})$
- $\sup_{\alpha \in \mathbb{R}_+} c_p(\alpha s_p, s_{-p}) = +\infty$
- $c_p(\cdot, s_{-p})$ has compact level sets (see Corollary 8.3.2 and Theorem 8.4 in [28])
- if $h_p \in \mathbb{R}_{++} s_p$, then

$$c_p'(s_p, s_{-p}; h_p) > 0 \text{ and } u_p'(s_p, s_{-p}; h_p) > 0.$$

[1]This assumption has no bite if s_p belongs to the relative interior of $\mathcal{S}_p$.

We denote the joint set of strategies and player p's feasible directions by

$$\mathcal{D}_p := \left\{(s_p, s_{-p}, h_p) \in \mathcal{S}_p \times \mathcal{S}_{-p} \times \mathbb{R}^{n_p} : h_p \in \mathcal{R}_{\mathcal{S}_p}(s_p)\right\}.$$

Clearly, the problem of power optimization in a MIMO interference system can be regarded as a cost minimization game by setting, for all $p \in \mathcal{P} = \{1, \ldots, P\}$,

$$\mathcal{S}_p = \mathcal{H}_+ \quad (\mathcal{H}_+ \subset \mathbb{R}^{d_p} \text{ for some finite integer } d_p \geq 1) \quad (3)$$

and, for all $(Q_p, Q_{-p}) \in \mathcal{S}_p \times \mathcal{S}_{-p}$,

$$c_p(Q_p, Q_{-p}) = \mathrm{trace}(Q_p) \quad (4)$$

$$u_p(Q_p, Q_{-p}) = \log_2 \det\left(I + R_p^{-1/2} H_{p,p} Q_p H_{p,p}^{\dagger} R_p^{-1/2}\right) \quad (5)$$

where $R_p = I + \sum_{q \neq p} H_{p,q} Q_q H_{p,q}^{\dagger}$. We note that Assumption 1 holds in this cost minimization game (if any channel matrix $H_{p,p}$ is rank deficient, then the set of strategies need to be restricted in such a way that part (iv) of Assumption 1 is satisfied).

Given the setup above, the objective of each player $p \in \mathcal{P}$ is to choose a strategy $s_p \in \mathcal{S}_p$ with minimal cost while achieving a certain utility level $\bar{u}_p \geq 0$. More precisely, each player $p \in \mathcal{P}$ is to solve the following cost minimization problem for some target utility level $\bar{u}_p \geq 0$:

$$\inf_{s_p \in \mathcal{C}_p(s_{-p})} c_p(s_p, s_{-p}) \quad (6)$$

where

$$\mathcal{C}_p(s_{-p}) := \{\check{s}_p \in \mathcal{S}_p : u_p(\check{s}_p, s_{-p}) \geq \bar{u}_p\}$$

(without the knowledge of $s_{-p} \in \mathcal{S}_{-p}$, in actuality!). We will refer to this cost minimization game corresponding to the target utility levels $\bar{u} \in \mathbb{R}_+^P$ as $\Gamma^c(\bar{u})$.

Note that, for any $s_{-p} \in \mathcal{S}_{-p}$, $\mathcal{C}_p(s_{-p})$ is closed and convex; moreover, if $\mathcal{C}_p(s_{-p})$ is nonempty, then it is *unbounded*, i.e., for any $\alpha \in \mathbb{R}_+$, there is an $s_p \in \mathcal{C}_p(s_{-p})$ such that $\|s_p\| \geq \alpha$.

Lemma 1: For any $s_{-p} \in \mathcal{S}_{-p}$, if $\mathcal{C}_p(s_{-p}) \neq \emptyset$, then the infimum in (6) is achieved by some $\hat{s}_p \in \mathcal{C}_p(s_{-p})$ such that $u_p(\hat{s}_p, s_{-p}) = \bar{u}_p$.

Proof: Pick any $\check{s}_p \in \mathcal{C}_p(s_{-p})$ and let $\check{c}_p := c_p(\check{s}_p, s_{-p})$. The infimum in (6) equals

$$\inf\left\{c_p(s_p, s_{-p}) : s_p \in \mathcal{C}_p(s_{-p}),\ c_p(s_p, s_{-p}) \leq \check{c}_p\right\}.$$

Since $c_p(\cdot, s_{-p})$ has compact level sets, the set

$$\{s_p \in \mathcal{C}_p(s_{-p}) : c_p(s_p, s_{-p}) \leq \check{c}_p\}$$

is compact. In addition, $c_p(\cdot, s_{-p})$ is continuous in $\mathcal{S}_p$ by assumption. Therefore, there exists some $\hat{s}_p \in \mathcal{C}_p(s_{-p})$ achieving the infimum above as well as the infimum in (6). If $u_p(\hat{s}_p, s_{-p}) > \bar{u}_p$, then there exists some $\alpha \in (0, 1)$ such that $u_p(\alpha\hat{s}_p, s_{-p}) \geq \bar{u}_p$ and $c_p(\alpha\hat{s}_p, s_{-p}) < c_p(\hat{s}_p, s_{-p})$, which contradicts the optimality of $\hat{s}_p$. Therefore, $u_p(\hat{s}_p, s_{-p}) = \bar{u}_p$. ■

Regarding each player p's problem (6), we will point out the following facts for future reference. For any $s_{-p} \in \mathcal{S}_{-p}$, each player p's problem is an *ordinary convex program*; see [28]. If $s_{-p} \in \mathcal{S}_{-p}$ is such that $\mathcal{C}_p(s_{-p}) \neq \emptyset$, then, by Lemma 1, some $\hat{s}_p \in \mathcal{C}_p(s_{-p})$ solves player p's problem (6) with $u_p(\hat{s}_p, s_{-p}) = \bar{u}_p$. This implies that, for some $\bar{s}_p \in \mathcal{S}_p$, the regularity condition $\bar{u}_p - u_p(\bar{s}_p, s_{-p}) < 0$ is satisfied. Hence, by Corollary 28.2.1 in [28], there exists a multiplier $\lambda_p(s_{-p}) \in \mathbb{R}_+$, called a Kuhn-Tucker multiplier, such that

$$\begin{aligned} c_p(\hat{s}_p, s_{-p}) &= \min_{s_p \in \mathcal{C}_p(s_{-p})} c_p(s_p, s_{-p}) \\ &= \min_{s_p \in \mathcal{S}_p} c_p(s_p, s_{-p}) + \lambda_p(s_{-p})\left(\bar{u}_p - u_p(s_p, s_{-p})\right). \end{aligned}$$

Furthermore, by Theorem 28.3 in [28], the Kuhn-Tucker condition

$$c_p'(\hat{s}_p, s_{-p}; h_p) - \lambda_p(s_{-p}) u_p'(\hat{s}_p, s_{-p}; h_p) \geq 0 \quad (7)$$

must be satisfied for all $h_p \in \mathcal{R}_{\mathcal{S}_p}(\hat{s}_p)$.

A. Equilibrium

A profile of strategies $s^* \in \mathcal{S}$ that mutually solves each player's cost minimization problem (6) is called a *generalized Nash equilibrium*[2]. In other words, a profile of strategies $s^* \in \mathcal{S}$ is an equilibrium if and only if, for all $p \in \mathcal{P}$,

$$c_p(s_p^*, s_{-p}^*) = \min_{s_p \in \mathcal{C}_p(s_{-p}^*)} c_p(s_p, s_{-p}^*).$$

An equilibrium can also be regarded as a fixed point of the best response correspondence $BR^c : \mathcal{S} \rightrightarrows \mathcal{S}$, where $BR^c = (BR_1^c, \ldots, BR_P^c)$ is defined by[3]: for all $p \in \mathcal{P}$ and $s_{-p} \in \mathcal{S}_{-p}$,

$$BR_p^c(s_{-p}) := \operatorname*{argmin}_{s_p \in \mathcal{C}_p(s_{-p})} c_p(s_p, s_{-p}).$$

With this notation, a profile of strategies $s^* \in \mathcal{S}$ constitutes an equilibrium if and only if

$$s^* \in BR^c(s^*).$$

At equilibrium, no player has an incentive to unilaterally deviate to an alternative strategy. Hence, the concept of equilibrium is quite relevant in situations where optimizing the overall system is not feasible. In the context of power optimization in MIMO interference systems, a group of selfish links interested in minimizing their own power consumptions may settle only at an equilibrium. Hence, it is of interest to study the properties of equilibrium, starting with its existence.

B. Existence of Equilibrium

A widely-used method for showing the existence of equilibrium in noncooperative games is to use the various fixed point theorems available in the literature. A specialization of an existence result from the literature, namely Theorem 4.3.1 in [29], which relies on Kakutani's fixed point theorem, is given below using our own notation.

Theorem 1 (A specialization of Theorem 4.3.1 in [29]): Consider a game with the finite set $\mathcal{P} = \{1, ..., P\}$ of players in which each player $p \in \mathcal{P}$ chooses $x_p \in \mathcal{X}_p$ to solve

$$\max_{x_p \in \mathcal{F}_p(x_{-p})} f_p(x_p, x_{-p})$$

[2] In the case of fixed strategy sets, the term Nash equilibrium, as opposed to generalized Nash equilibrium, is used. For simplicity, we henceforth refer to a generalized Nash equilibrium simply as an equilibrium.

[3] We suppress the dependence of BR^c on $\bar{u}$ to avoid cluttering the notation.

where

(i) $\mathcal{X}_p$ is a nonempty compact convex subset of $\mathbb{R}^{d_p}$ for some finite integer $d_p \geq 1$

(ii) $\mathcal{F}_p : \mathcal{X}_{-p} \rightrightarrows \mathcal{X}_p$ is both upper semi-continuous[4] (u.s.c.) and lower semi-continuous[5] (l.s.c.) in $\mathcal{X}_{-p} := \bigtimes_{q \in \mathcal{P}-\{p\}} \mathcal{X}_q$

(iii) for all $x_{-p} \in \mathcal{X}_{-p}$, $\mathcal{F}_p(x_{-p})$ is nonempty, closed, and convex

(iv) $f_p : \mathcal{X} \mapsto \mathbb{R}$ is continuous in $\mathcal{X} := \bigtimes_{p \in \mathcal{P}} \mathcal{X}_p$

(v) for all $x_{-p} \in \mathcal{X}_{-p}$, $f_p(\cdot, x_{-p})$ is quasi-concave in $\mathcal{X}_p$.

The game described above possesses an equilibrium.

Remark 1: If, for all $p \in \mathcal{P}$, $\mathcal{F}_p \equiv \mathcal{A}_p$ where $\mathcal{A}_p$ is some nonempty closed convex subset of $\mathcal{X}_p$, then condition (ii) and (iii) of Theorem 1 are satisfied.

Remark 2: Let $\mathcal{X}$ be as in Theorem 1 and consider a function $g_p : \mathcal{X} \mapsto \mathbb{R}$ that satisfies: (a) g_p is continuous in $\mathcal{X}$, (b) for each $x_{-p} \in \mathcal{X}_{-p}$, $g_p(\cdot, x_{-p})$ is convex in $\mathcal{X}_p$, (c) for each $x_{-p} \in \mathcal{X}_{-p}$, there exists some $\bar{x}_p \in \mathcal{X}_p$ such that $g_p(\bar{x}_p, x_{-p}) < 0$. If, for all $x_{-p} \in \mathcal{X}_{-p}$, $\mathcal{F}_p(x_{-p}) = \{x_p \in \mathcal{X}_p : g_p(x_p, x_{-p}) \leq 0\}$, then $\mathcal{F}_p$ satisfies condition (ii) and (iii) of Theorem 1. To see the u.s.c. property, we note that the graph of $\mathcal{F}_p$, $\{(x_{-p}, x_p) \in \mathcal{X}_{-p} \times \mathcal{X}_p : g_p(x_p, x_{-p}) \leq 0\}$, is closed in $\mathcal{X}_{-p} \times \mathcal{X}_p$. This together with the compactness of $\mathcal{X}_p$ implies that $\mathcal{F}_p$ is u.s.c. in $\mathcal{X}_{-p}$; see Theorem 2.2.3 in [29]. The l.s.c. property follows from the assumed properties of g_p, $\mathcal{X}_p$, $\mathcal{X}_{-p}$; see Theorem 12 in [30]. The assumed properties of g_p also implies condition (iii) of Theorem 1.

It is tempting to apply Theorem 1 to the cost minimization game $\Gamma^c(\bar{u})$ by setting

$$\mathcal{X}_p = \mathcal{S}_p, \quad f_p = -c_p, \quad \mathcal{F}_p = \mathcal{C}_p, \quad \text{for all } p \in \mathcal{P}.$$

The main difficulty is that condition (i) of Theorem 1 is not satisfied, because $\mathcal{S}_1, \ldots, \mathcal{S}_P$ are not compact (although they are nonempty, closed, and convex, by assumption). This difficulty, namely the unboundedness of $\mathcal{S}_1, \ldots, \mathcal{S}_P$, can be circumvented, if there are nonempty compact convex subsets $\bar{\mathcal{S}}_p \subset \mathcal{S}_p$, for all $p \in \mathcal{P}$, such that $\bar{\mathcal{S}} := \bigtimes_{p \in \mathcal{P}} \bar{\mathcal{S}}_p$ is stable under BR^c, i.e.,

$$BR^c(\bar{\mathcal{S}}) := \{s \in \mathcal{S} : s \in BR^c(\bar{s}), \bar{s} \in \bar{\mathcal{S}}\} \subset \bar{\mathcal{S}}.$$

If this is indeed the case and the other conditions of Theorem 1 are satisfied, then the restriction of $\Gamma^c(\bar{u})$ to $\bar{\mathcal{S}}$ would possess an equilibrium which would also be an equilibrium of $\Gamma^c(\bar{u})$.

However, finding such subsets $\bar{\mathcal{S}}_1, \ldots, \bar{\mathcal{S}}_P$ itself requires an equilibration process (i.e., a process leading to subsets $\bar{\mathcal{S}}_1, \ldots, \bar{\mathcal{S}}_P$ that are in "equilibrium" in some sense), which is not necessarily an easier task than establishing the existence of equilibrium. For instance, there is no obvious way of accomplishing this for the problem of power optimization in a general MIMO interference system. To overcome this obstacle, we explore a duality relation with a utility maximization game introduced in the next section.

[4] $\mathcal{F}_p$ is called upper semi-continuous if for every sequence $\{x_{-p}^n\}_n$ in $\mathcal{X}_{-p}$ converging to an arbitrary $\bar{x}_{-p}$, and for every neighborhood $\mathcal{G}$ of $\mathcal{F}_p(\bar{x}_{-p})$ in $\mathcal{X}_p$, there exists n_0 such that $\mathcal{F}_p(x_{-p}^n) \subset \mathcal{G}$ for all $n \geq n_0$.

[5] $\mathcal{F}_p$ is called lower semi-continuous if for every sequence $\{x_{-p}^n\}_n$ in $\mathcal{X}_{-p}$ converging to an arbitrary $\bar{x}_{-p}$, and for every open subset $\mathcal{G}$ of $\mathcal{X}_p$ for which $\mathcal{F}_p(\bar{s}_{-p}) \cap \mathcal{G} \neq \emptyset$, there exists n_0 such that $\mathcal{F}_p(x_{-p}^n) \cap \mathcal{G} \neq \emptyset$ for all $n \geq n_0$.

IV. A UTILITY MAXIMIZATION GAME

Using the same setup as in the previous section, a utility maximization game is introduced as a noncooperative game in which each player $p \in \mathcal{P}$ is to maximize its utility while keeping its cost below a certain level. More precisely, each player $p \in \mathcal{P}$ is to solve the following utility maximization problem for some target cost level $\bar{c}_p \geq 0$:

$$\max_{s_p \in \mathcal{U}_p(s_{-p})} u_p(s_p, s_{-p}) \tag{8}$$

where $\mathcal{U}_p(s_{-p}) := \{\check{s}_p \in \mathcal{S}_p : c_p(\check{s}_p, s_{-p}) \leq \bar{c}_p\}$. Note that $\mathcal{U}_p(s_{-p})$ is nonempty, compact, and convex. Hence, the maximum in (8) is always achieved by some $\hat{s}_p \in \mathcal{U}_p(s_{-p})$ such that $c_p(\hat{s}_p, s_{-p}) = \bar{c}_p$.

We will refer to this utility maximization game corresponding to the target cost levels $\bar{c} \in \mathbb{R}_+^P$ as $\Gamma^u(\bar{c})$. The concept of equilibrium and the best response correspondence BR^u for $\Gamma^u(\bar{c})$ are defined in a completely analogous way as in the case of the cost minimization game $\Gamma^c(\bar{u})$.

A. A Duality Relation

The relevance of the utility maximization game in the context of this paper is due to the following duality relation.

Proposition 1: Fix $\bar{s} \in \mathcal{S}$, and let

$$\bar{c} := (c_1(\bar{s}), \ldots, c_P(\bar{s})) \quad \text{and} \quad \bar{u} := (u_1(\bar{s}), \ldots, u_P(\bar{s})).$$

Let $\mathcal{E}^c(\bar{u})$, $\mathcal{E}^u(\bar{c})$ denote the sets of equilibria for the games $\Gamma^c(\bar{u})$, $\Gamma^u(\bar{c})$, respectively. Then,

$$\bar{s} \in \mathcal{E}^c(\bar{u}) \qquad \Leftrightarrow \qquad \bar{s} \in \mathcal{E}^u(\bar{c}).$$

Proof: Suppose that $\bar{s} \notin \mathcal{E}^u(\bar{c})$. Hence, for some $p \in \mathcal{P}$, there exists an $\hat{s}_p \in \mathcal{S}_p$ such that

$$u_p(\hat{s}_p, \bar{s}_{-p}) > u_p(\bar{s}_p, \bar{s}_{-p}) \quad \text{and} \quad c_p(\hat{s}_p, \bar{s}_{-p}) \leq \bar{c}_p.$$

Because of the strict inequality above, we must have $\hat{s}_p \neq 0$. Hence, for some $\alpha \in (0, 1)$,

$$c_p(\alpha \hat{s}_p, \bar{s}_{-p}) < c_p(\bar{s}_p, \bar{s}_{-p}) \quad \text{and} \quad u_p(\alpha \hat{s}_p, \bar{s}_{-p}) \geq \bar{u}_p$$

which means that $\bar{s} \notin \mathcal{E}^c(\bar{u})$. Therefore, $\bar{s} \in \mathcal{E}^c(\bar{u}) \Rightarrow \bar{s} \in \mathcal{E}^u(\bar{c})$. The proof of the reversed implication is similar. ∎

This duality relation reveals that the cost minimization game $\Gamma^c(\bar{u})$ possesses equilibrium if and only if the target utility levels $\bar{u}$ can be achieved at an equilibrium of the utility maximization game $\Gamma^u(\bar{c})$ for some target cost levels $\bar{c}$. Therefore, it is of interest to characterize the set of equilibrium utility levels defined as

$$\mathbb{U}_e := \left\{(u_1(\bar{s}), \ldots, u_P(\bar{s})) \in \mathbb{R}_+^P : \bar{s} \in \mathcal{E}^u(\bar{c}), \bar{c} \in \mathbb{R}_+^P\right\}. \tag{9}$$

Prior to characterizing $\mathbb{U}_e$, however, we first address the issue of the existence of equilibrium in the utility maximization game.

B. Existence of Equilibrium

Applying Theorem 1 to the utility minimization game $\Gamma^u(\bar{c})$ to establish the existence of equilibrium results in the same difficulty as in the case of the cost minimization game $\Gamma^c(\bar{u})$, that is, $\mathcal{S}_1, \ldots, \mathcal{S}_P$ are not compact. Similarly, the unboundedness of $\mathcal{S}_1, \ldots, \mathcal{S}_P$, can be circumvented, if there are nonempty compact convex subsets $\bar{\mathcal{S}}_p \subset \mathcal{S}_p$, for all $p \in \mathcal{P}$, such that $\bar{\mathcal{S}} := \bigtimes_{p\in\mathcal{P}} \bar{\mathcal{S}}_p$ is stable under BR^u, i.e.,

$$BR^u(\bar{\mathcal{S}}) := \{s \in \mathcal{S} : s \in BR^u(\bar{s}), \bar{s} \in \bar{\mathcal{S}}\} \subset \bar{\mathcal{S}}.$$

It turns out that the existence of such subsets $\bar{\mathcal{S}}_1, \ldots, \bar{\mathcal{S}}_P$ can be shown.

Lemma 2: For all $p \in \mathcal{P}$, $\bar{c}_p \in \mathbb{R}_+$, the set $\bar{\mathcal{S}}_p$ defined as

$$\begin{aligned} \bar{\mathcal{S}}_p &:= \mathrm{cc}\left(\mathcal{U}_p\left(\mathcal{S}_{-p}\right)\right) \\ &= \mathrm{cc}\left(\{s_p \in \mathcal{S}_p : c_p(s_p, s_{-p}) \le \bar{c}_p, s_{-p} \in \mathcal{S}_{-p}\}\right) \end{aligned} \tag{10}$$

is nonempty, compact, and convex.

Proof: It suffices to show that, for all $p \in \mathcal{P}$, $\mathcal{U}_p\left(\mathcal{S}_{-p}\right)$ is bounded. Suppose that, for some $p \in \mathcal{P}$, $\mathcal{U}_p\left(\mathcal{S}_{-p}\right)$ is unbounded. Then, there must exist a sequence $\{s^n\}_{n\ge1}$ in $\mathcal{S}$ such that,

$$c_p(s^n) \le \bar{c}_p \quad \text{and} \quad \|s_p^n\| \ge n.$$

Let $M \ge 0$ be arbitrary. Since $c_p(\cdot, s_{-p}^n)$ is convex in $\mathcal{S}_p$, we have, for all $n \ge M$,

$$0 \le c_p\left(s_p^n \frac{M}{\|s_p^n\|}, s_{-p}^n\right) \le \frac{M}{\|s_p^n\|} c_p(s^n) \le \frac{M}{n}\bar{c}_p.$$

This implies that

$$\inf_{\hat{s}\in\mathcal{S}:\|\hat{s}_p\|\ge M} c_p(\hat{s}) = 0$$

which contradicts part (v) of Assumption 1. ■

This leads us to the following result.

Proposition 2: For any $\bar{c} \in \mathbb{R}_+^P$, the utility maximization game $\Gamma^u(\bar{c})$ possesses an equilibrium.

Proof: If $\bar{c}_p = 0$, for any $p \in \mathcal{P}$, then $BR_p^u(\mathcal{S}_{-p}) = \mathcal{U}_p(\mathcal{S}_{-p}) = \{0\}$. Therefore, we can remove any such player $p \in \mathcal{P}$ with $\bar{c}_p = 0$ by substituting 0 into s_p throughout and obtaining a reduced utility maximization game with fewer players. Hence, we only consider the case where $\bar{c} \in \mathbb{R}_{++}^P$, without loss of generality.

Let $\bar{\mathcal{S}}_p$ be as in (10), for all $p \in \mathcal{P}$. Let $\bar{\mathcal{S}} := \bigtimes_{p\in\mathcal{P}} \bar{\mathcal{S}}_p$, $\bar{\mathcal{S}}_{-p} := \bigtimes_{q\in\mathcal{P}-\{p\}} \bar{\mathcal{S}}_q$. By construction, we have $BR^u(\bar{\mathcal{S}}) \subset \bar{\mathcal{S}}$. Consider the restriction[6] $\Gamma^u(\bar{c})|_{\bar{\mathcal{S}}}$ of $\Gamma^u(\bar{c})$ to $\bar{\mathcal{S}}$. An equilibrium of $\Gamma^u(\bar{c})|_{\bar{\mathcal{S}}}$, if exists, is also an equilibrium of $\Gamma^u(\bar{c})$.

We now apply Theorem 1 to $\Gamma^u(\bar{c})|_{\bar{\mathcal{S}}}$ by letting

$$\mathcal{X}_p = \bar{\mathcal{S}}_p, \quad f_p = u_p|_{\bar{\mathcal{S}}}, \quad \mathcal{F}_p = \mathcal{U}_p|_{\bar{\mathcal{S}}_{-p}}, \quad \text{for all } p \in \mathcal{P}.$$

Since $\bar{\mathcal{S}}_p$ is a nonempty compact convex subset of $\mathbb{R}^{d_p}$, for all $p \in \mathcal{P}$, condition (i) of Theorem 1 is satisfied. Note that, for each $s_{-p} \in \bar{\mathcal{S}}_{-p}$, $\mathcal{U}_p|_{\bar{\mathcal{S}}_{-p}}(s_{-p}) = \{s_p \in \bar{\mathcal{S}}_p : c_p(s_p, s_{-p}) \le \bar{c}_p\}$, which implies that $\mathcal{U}_p|_{\bar{\mathcal{S}}_{-p}}$ is both u.s.c. and l.s.c. in $\bar{\mathcal{S}}_{-p}$; see Remark 2. Thus, condition (ii) of Theorem 1 is also satisfied. Finally, conditions (iii), (iv), and (v) of Theorem 1 are readily satisfied due to Assumption 1. Hence, $\Gamma^u(\bar{c})|_{\bar{\mathcal{S}}}$ possesses equilibrium. ■

[6] The restriction $\Gamma^u(\bar{c})|_{\bar{\mathcal{S}}}$ is the same as $\Gamma^u(\bar{c})$ except that the players are allowed choose their joint strategies only in $\bar{\mathcal{S}}$.

C. Equilibrium versus Achievable Utility Levels

We now deal with the issue of characterizing the set of equilibrium utility levels $\mathbb{U}_e$ defined in (9). Clearly, $\mathbb{U}_e \subset \mathbb{U}_a$ where $\mathbb{U}_a$ denotes the set of achievable utility levels, i.e.,

$$\mathbb{U}_a := \{(u_1(s), \ldots, u_P(s)) : s \in \mathcal{S}\}.$$

The next proposition shows a simple case where $\mathbb{U}_e = \mathbb{U}_a$.

Proposition 3: If, for all $p \in \mathcal{P}$, there exists $\hat{s}_p \in \mathcal{S}_p$ such that $\mathcal{S}_p = \mathbb{R}_+\hat{s}_p$, then $\mathbb{U}_e = \mathbb{U}_a$.

Proof: Fix $\bar{s} \in \mathcal{S}$, and let $\bar{u} := (u_1(\bar{s}), \ldots, u_P(\bar{s}))$, $\bar{c} := (c_1(\bar{s}), \ldots, c_P(\bar{s}))$. Then, $\bar{s} \in \mathcal{E}^u(\bar{c})$. To see this, consider the problem

$$\max_{s_p\in\mathcal{S}_p:c_p(s_p,\bar{s}_{-p})\le\bar{c}_p} u_p(s_p, \bar{s}_{-p}).$$

Clearly, the maximum above is uniquely achieved by $\bar{s}_p$. ■

In general, however, $\mathbb{U}_e$ is a proper subset of $\mathbb{U}_a$, i.e., $\mathbb{U}_e \subsetneq \mathbb{U}_a$.

Example 1: For some $\beta \in (0, 1)$, and for all $p \in \mathcal{P} = \{1, 2\}$, let

$$\begin{aligned} &\mathcal{S}_p = \mathbb{R}_+^2 \\ &c_p(s_p, s_{-p}) = s_{p,1} + s_{p,2} \\ &u_p(s_p, s_{-p}) = \sum_{k=1}^{2} \log_2\left(1 + \frac{s_{p,k}}{1 + \beta s_{-p,k}}\right). \end{aligned}$$

Any target utility level $\bar{u} \in \mathbb{R}_+^2$ can be achieved by $(s_1, s_2) = ((2^{\bar{u}_1} - 1, 0), (0, 2^{\bar{u}_2} - 1))$, hence $\mathbb{U}_a = \mathbb{R}_+^2$. The unique equilibrium of $\Gamma^u(\bar{c})$ for any $\bar{c} \in \mathbb{R}_+^2$ is $((\bar{c}_1/2, \bar{c}_1/2), (\bar{c}_2/2, \bar{c}_2/2))$ which leads to utility levels

$$\left(\log_2\left(1 + \frac{\bar{c}_1/2}{1 + \beta\bar{c}_2/2}\right)^2, \log_2\left(1 + \frac{\bar{c}_2/2}{1 + \beta\bar{c}_1/2}\right)^2\right).$$

This implies that

$$\mathbb{U}_e = \left\{\bar{u} \in \mathbb{R}_+^2 : (\sqrt{2^{\bar{u}_1}} - 1)(\sqrt{2^{\bar{u}_2}} - 1) < 1/\beta^2\right\}.$$

Figure 1 shows $\mathbb{U}_e$ for $\beta = 1/2$.

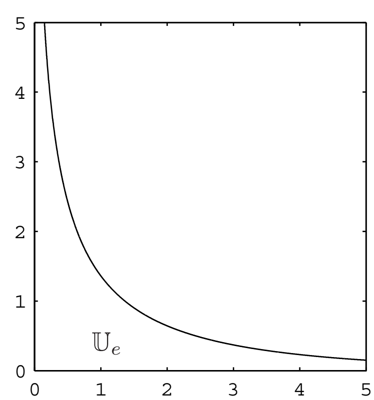

Fig. 1. The set of equilibrium utility levels in Example 1 for $\beta = 1/2$.

V. THE CASE OF DECOUPLED COST FUNCTIONS

In this section, we consider the special case of decoupled cost functions where each player's cost function depends only on its own strategy. We assume that, for all $p \in \mathcal{P}$, $(s_p, s_{-p}) \in \mathcal{S}$,

$$c_p(s_p, s_{-p}) = c_p(s_p)$$

where, by a slight abuse of notation, $c_p(s_p)$ denotes player p's cost for using the resource $s_p \in \mathcal{S}_p$ regardless of the strategies of the other players. The problem of power optimization in a MIMO interference system falls into this special case.

In the case of decoupled cost functions, it is possible to obtain an inner estimate of the set of equilibrium utility levels, without resorting to an equilibration process. For this, we define the set of minimax utility levels as

$$\mathbb{U}_m := \bigcup_{\bar{c} \in \mathbb{R}_+^P} \Big\{ \bar{u} \in \mathbb{R}_+^P : \text{for all } p \in \mathcal{P}, \\ \bar{u}_p < \min_{s_{-p} \in \bar{\mathcal{S}}_{-p}(\bar{c}_{-p})} \max_{s_p \in \bar{\mathcal{S}}_p(\bar{c}_p)} u_p(s_p, s_{-p}) \Big\} \quad (11)$$

where, for all $p \in \mathcal{P}$, $\bar{\mathcal{S}}_p(\bar{c}_p) = \{s_p \in \mathcal{S}_p : c_p(s_p) \leq \bar{c}_p\}$, and $\bar{\mathcal{S}}_{-p}(\bar{c}_{-p}) := \bigtimes_{q \in \mathcal{P} - \{p\}} \bar{\mathcal{S}}_q(\bar{c}_q)$. Note that, for all $\bar{c}_p \geq 0$, $\bar{\mathcal{S}}_p(\bar{c}_p)$ is nonempty, compact, and convex; and the minimum in (11) exists due to the Maximum Theorem; see Theorem 2.3.1 in [29]. Furthermore, $\mathbb{U}_m \neq \emptyset$, since the right-hand side of the strict inequality in (11) is always strictly positive, for any $\bar{c} \in \mathbb{R}_{++}^P$.

Proposition 4: In the case of decoupled cost functions,

$$\mathbb{U}_m \subset \mathbb{U}_e.$$

Proof: Let $(\bar{u}, \bar{c}) \in \mathbb{U}_m \times \mathbb{R}_+^P$ be such that, for all $p \in \mathcal{P}$,

$$\bar{u}_p < \min_{s_{-p} \in \bar{\mathcal{S}}_{-p}(\bar{c}_{-p})} \max_{s_p \in \bar{\mathcal{S}}_p(\bar{c}_p)} u_p(s_p, s_{-p}).$$

This means that, for all $p \in \mathcal{P}$, $s_{-p} \in \bar{\mathcal{S}}_{-p}(\bar{c}_{-p})$, there exists some $\check{s}_p \in \bar{\mathcal{S}}_p(\bar{c}_p)$ (which depends on s_{-p}) such that

$$\bar{u}_p < u_p(\check{s}_p, s_{-p}) \qquad \text{and} \qquad c_p(\check{s}_p) \leq \bar{c}_p. \quad (12)$$

Consider the game $\Gamma^c(\bar{u})$. From (12), it is straightforward to see that

$$\begin{aligned} s_{-p} \in \bar{\mathcal{S}}_{-p}(\bar{c}_{-p}), \hat{s}_p \in BR_p^c(s_{-p}) \quad &\Rightarrow \quad c_p(\hat{s}_p) \leq \bar{c}_p \\ &\Rightarrow \quad \hat{s}_p \in \bar{\mathcal{S}}_p(\bar{c}_p). \end{aligned}$$

Therefore, $\bar{\mathcal{S}}(\bar{c}) := \bigtimes_{p \in \mathcal{P}} \bar{\mathcal{S}}_p(\bar{c}_p)$ is stable under BR^c. Applying Theorem 1 to the restriction $\Gamma^c(\bar{u})|_{\bar{\mathcal{S}}(\bar{c})}$ of $\Gamma^c(\bar{u})$ to $\bar{\mathcal{S}}(\bar{c})$ leads to the existence of equilibrium in $\Gamma^c(\bar{u})|_{\bar{\mathcal{S}}(\bar{c})}$ and $\Gamma^c(\bar{u})$; see Remark 2. Therefore, $\bar{u} \in \mathbb{U}_e$. ■

An immediate consequence of Proposition 4 is that all target utility levels that are small enough are minimax (hence equilibrium) utility levels.

Proposition 5: If the player cost functions are decoupled, then there exists some $\hat{u} \in \mathbb{R}_{++}^P$ such that

$$\{\bar{u} \in \mathbb{R}_+^P : \bar{u} \leq \hat{u}\} \subset \mathbb{U}_m.$$

Proof: It readily follows from the definition of $\mathbb{U}_m$. ■

In some cases, it is possible to obtain the entire set of equilibrium utility levels through the set of minimax utility levels, i.e., $\mathbb{U}_m = \mathbb{U}_e$. However, in general, $\mathbb{U}_m$ can be a proper set of $\mathbb{U}_e$, i.e., $\mathbb{U}_m \subsetneq \mathbb{U}_e$.

Example 2: Let us reconsider the setup in Example 1. We compute $\mathbb{U}_m$ as: for all $\beta \geq 0$,

$$\mathbb{U}_m = \left\{ \bar{u} \in \mathbb{R}_+^2 : (\sqrt{2^{\bar{u}_1}} - 1)(\sqrt{2^{\bar{u}_2}} - 1) < 1/\beta^2 \right\}.$$

This implies that, for any $\beta \in (0, 1)$, $\mathbb{U}_m = \mathbb{U}_e$. In the case of $\beta \geq 1$, however, $\Gamma^u(\bar{c})$ for any $\bar{c} \in \mathbb{R}_+^2$ possesses multiple equilibria, one of which is $((\bar{c}_1/2, \bar{c}_1/2), (\bar{c}_2/2, \bar{c}_2/2))$. For example, in the case where $\beta \geq 1$ and $\bar{c}_1 = \bar{c}_2 = \rho$, for some $\rho > 0$, $((\rho, 0), (0, \rho))$ is an additional equilibrium of $\Gamma^u(\bar{c})$ with the corresponding equilibrium utility levels $(\log_2(1+\rho), \log_2(1+\rho))$. If ρ is large enough, then $(\log_2(1+\rho), \log_2(1+\rho)) \notin \mathbb{U}_m$. Therefore, for any $\beta \geq 1$, we have $\mathbb{U}_m \subsetneq \mathbb{U}_e$.

A. An Application to Power Minimization in MIMO Systems

To demonstrate the benefits of the minimax approach, we now present an example motivated by the problem of power optimization in MIMO interference systems for the case of diagonal channel matrices; see [19].

Example 3: For all $p \in \mathcal{P} = \{1, \ldots, P\}$, let $\mathcal{S}_p = \mathbb{R}_+^d$, for some finite integer $d \geq 1$, $c_p(s_p) = \sum_{k=1}^d s_{p,k}$ and let $u_p(s_p, s_{-p})$ be given by

$$\sum_{k=1}^d \ln \left(1 + \frac{|h_{p,p,k}|^2 s_{p,k}}{(\sigma_{p,k})^2 + \sum_{q \in \mathcal{P} - \{p\}} |h_{p,q,k}|^2 s_{q,k}} \right)$$

where $h_{p,q,k} \neq 0$, for all $p, q \in \mathcal{P}$, $k \in \{1, \ldots, d\}$, are given complex scalars, and $\sigma_{p,k} > 0$, for all $p \in \mathcal{P}$, $k \in \{1, \ldots, d\}$, are given real scalars.

To estimate $\mathbb{U}_m$, let, for some $\bar{c} \in \mathbb{R}_+^P$, for all $p \in \mathcal{P}$,

$$\bar{\mathcal{S}}_p(\bar{c}_p) := \left\{ s_p \in \mathbb{R}_+^d : \sum_{k=1}^d s_{p,k} \leq \bar{c}_p \right\}.$$

By evaluating $u_p(s_p, s_{-p})$ at each s_p of the form

$$(0, \ldots, 0, \bar{c}_p, 0, \ldots, 0)$$

we obtain: for all $p \in \mathcal{P}$, $k \in \{1, \ldots, d\}$,

$$\begin{aligned} &\min_{s_{-p} \in \bar{\mathcal{S}}_{-p}(\bar{c}_{-p})} \max_{s_p \in \bar{\mathcal{S}}_p(\bar{c}_p)} u_p(s_p, s_{-p}) \geq \quad (13) \\ &\min_{s_{-p} \in \bar{\mathcal{S}}_{-p}(\bar{c}_{-p})} \ln \left(1 + \frac{|h_{p,p,k}|^2 \bar{c}_p}{(\sigma_{p,k})^2 + \sum_{q \in \mathcal{P} - \{p\}} |h_{p,q,k}|^2 s_{q,k}} \right) \\ &= \ln \left(1 + \frac{|h_{p,p,k}|^2 \bar{c}_p}{(\sigma_{p,k})^2 + \sum_{q \in \mathcal{P} - \{p\}} |h_{p,q,k}|^2 \bar{c}_q} \right). \end{aligned}$$

Hence, for any $\bar{u} \in \mathbb{R}_+^P$, if there exists some $\bar{c} \in \mathbb{R}_+^P$ and $(k_1, \ldots, k_P) \in \{1, \ldots, d\}^P$ such that, for all $p \in \mathcal{P}$,

$$\ln \left(1 + \frac{|h_{p,p,k_p}|^2 \bar{c}_p}{(\sigma_{p,k_p})^2 + \sum_{q \in \mathcal{P} - \{p\}} |h_{p,q,k_p}|^2 \bar{c}_q} \right) > \bar{u}_p$$

then $\bar{u} \in \mathbb{U}_m$. This condition can be rewritten as

$$Z_{k_1, \ldots, k_P}(\bar{u}) \bar{c} > b_{k_1, \ldots, k_P}(\bar{u})$$

where $Z_{k_1,\ldots,k_P}(\bar{u})$ is defined as

$$\begin{bmatrix} |h_{1,1,k_1}|^2 & -\gamma_1|h_{1,2,k_1}|^2 & \ldots & -\gamma_1|h_{1,P,k_1}|^2 \\ -\gamma_2|h_{2,1,k_2}|^2 & |h_{2,2,k_2}|^2 & \ldots & -\gamma_2|h_{2,P,k_2}|^2 \\ \vdots & \vdots & \ddots & \vdots \\ -\gamma_P|h_{P,1,k_P}|^2 & -\gamma_P|h_{P,2,k_P}|^2 & \ldots & |h_{P,P,k_P}|^2 \end{bmatrix}$$

and $b_{k_1,\ldots,k_P}(\bar{u})$ is defined as

$$\begin{bmatrix} \gamma_1(\sigma_{1,k_1})^2 & \gamma_2(\sigma_{2,k_2})^2 & \ldots & \gamma_P(\sigma_{P,k_P})^2 \end{bmatrix}^T$$

and $\gamma_p := e^{\bar{u}_p} - 1$, for all $p \in \mathcal{P}$. The matrix $Z_{k_1,\ldots,k_P}(\bar{u})$ has non-positive off diagonal entries. By (2.3) Theorem on page 134 of [31], $Z_{k_1,\ldots,k_P}(\bar{u})$ has an inverse with nonnegative entries if and only if $Z_{k_1,\ldots,k_P}(\bar{u})$ is a $P-$matrix, i.e., a matrix with positive principle minors[7]. Therefore, we have, for any $\bar{u} \in \mathbb{R}_+^P$,

$$\begin{aligned} &Z_{k_1,\ldots,k_P}(\bar{u}) \text{ is a } P\text{−matrix for some } (k_1,\ldots,k_P) \\ &\text{where } k_p \in \{1,\ldots,d\}, \text{ for all } p \in \mathcal{P} \\ &\qquad \Rightarrow \qquad \bar{u} \in \mathbb{U}_m. \end{aligned} \tag{14}$$

The main existence result (Theorem 5) of [19] states that, for any $\bar{u} \in \mathbb{R}_+^P$,

$$\begin{aligned} &Z_{k,\ldots,k}(\bar{u}) \text{ is a } P\text{−matrix, for all } k \in \{1,\ldots,d\} \\ &\qquad \Rightarrow \qquad \bar{u} \in \mathbb{U}_e. \end{aligned} \tag{15}$$

Thanks to the minimax approach, the condition in (14) obtained by very simple means relaxes the one in (15) obtained by using "an advanced degree-theoretic result for a nonlinear complementarity problem in order to handle the unboundedness of the users' rate constraints". Note that the condition in (14) can be relaxed further by tightening the lower bound in (13).

B. The Case of Weakly Coupled Utility Functions

As a final application of the minimax approach, we consider the case of weakly coupled utility functions (in the context of decoupled cost functions). We formalize the concept of weak coupling in terms of a coupling coefficient $\eta \geq 0$. For this, we introduce $\eta-$parameterizations of the cost minimization and the utility maximization games such that, for $\eta \geq 0$, each player $p \in \mathcal{P}$ incurs the cost $c_p(s_p)$ and receives the utility $u_p(s_p, \eta s_{-p})$ for the strategy profile $(s_p, s_{-p}) \in \mathcal{S}$. Thus, if $\eta = 0$, then both the utility and the cost functions are decoupled; whereas, if $\eta > 0$ is small, then the utility functions are weakly coupled and the cost functions are decoupled.

For instance, in the problem of power optimization in a MIMO interference system, the concept of weak coupling would correspond to weak interference at the receiver of each link caused by the other links. More precisely, if the channel matrices $H_{p,q}$, $p \neq q$, are replaced by $\sqrt{\eta}H_{p,q}$ where $\eta \geq 0$ is a parameter indicating the strength of the interference channels, then the utility function (i.e., the mutual information) for link p would take the form $u_p(Q_p, \eta Q_{-p})$; see (5). Hence, if $\eta > 0$ is small, then the utility functions $u_p(Q_p, \eta Q_{-p})$ would be weakly coupled; whereas, the cost functions $c_p(Q_p, Q_{-p}) = \text{trace}(Q_p)$ are decoupled.

We will refer to the achievable, minimax, and the equilibrium utility levels corresponding to η as $\mathbb{U}_a(\eta)$, $\mathbb{U}_m(\eta)$, and $\mathbb{U}_e(\eta)$, respectively. The following result states that essentially all utility levels achievable in the case of complete decoupling ($\eta = 0$) are minimax (hence equilibrium) utility levels in the case of weak coupling of the utility functions.

Proposition 6: If $0 \leq \bar{u} < \hat{u} \in \mathbb{U}_a(0)$, then there exists some $\bar{\eta} > 0$ such that, for all $\eta \in [0, \bar{\eta}]$,

$$\bar{u} \in \mathbb{U}_m(\eta) \subset \mathbb{U}_e(\eta).$$

Proof: Suppose that $0 \leq \bar{u} < \hat{u} \in \mathbb{U}_a(0)$. There exists some $\hat{s} \in \mathcal{S}$ such that, for all $p \in \mathcal{P}$,

$$\hat{u}_p = u_p(\hat{s}_p, 0).$$

For all $p \in \mathcal{P}$, let

$$\hat{c}_p := c_p(\hat{s}_p), \qquad \hat{\mathcal{S}}_p := \{s_p \in \mathcal{S}_p : c_p(s_p) \leq \hat{c}_p\},$$

and $\hat{S}_{-p} := \bigtimes_{q \in \mathcal{P}-\{p\}} \hat{\mathcal{S}}_q$. The subsets $\hat{S}_p$, for all $p \in \mathcal{P}$, are nonempty, compact, and convex. Due to the continuity of u_p, there exists some $\bar{\eta} > 0$ such that, for all $\eta \in [0, \bar{\eta}]$, $p \in \mathcal{P}$,

$$\bar{u}_p < \min_{s_{-p} \in \hat{\mathcal{S}}_{-p}} u_p(\hat{s}_p, \eta s_{-p})$$

which implies that

$$\bar{u}_p < \min_{s_{-p} \in \hat{\mathcal{S}}_{-p}} \max_{s_p \in \hat{\mathcal{S}}_p} u_p(s_p, \eta s_{-p}).$$

Therefore, for all $\eta \in [0, \bar{\eta}]$, $\bar{u} \in \mathbb{U}_m(\eta) \subset \mathbb{U}_e(\eta)$. ■

VI. AN EXACT PENALTY APPROACH

In the cost minimization problem, each player considers the achievement of a certain utility level as a hard constraint on itself. However, if the target utility levels are not equilibrium utility levels, then the players would not be able to agree on any resource allocation profile. In a realistic cost minimization game, a particular player would not know if its target utility level together with the other players' target utility levels constitute a profile of equilibrium utility levels. If the target utility levels are not equilibrium utility levels, then players may be caught up in an everlasting process of updating their strategies with no possibility of reaching an equilibrium solution; see Section VII on learning processes.

To alleviate this issue, we relax each player's hard constraint by incorporating a penalty term into each player's cost function, which penalizes the deviations from achieving its target utility level. More precisely, using the notation of Section III, we introduce a weighted cost minimization game, referred to as $\Gamma^w(\bar{u})$, in which each player $p \in \mathcal{P}$ is to solve

$$\min_{s_p \in \mathcal{S}_p} c_p(s_p, s_{-p}) + w_p\left(\bar{u}_p - u_p(s_p, s_{-p})\right)^+$$

where $w_p \geq 0$ is player p's unit cost of not achieving the target utility level $\bar{u}_p \geq 0$. It will be clear shortly that the minimum above always exists; see the proof of Proposition 7.

[7] A principal minor of a matrix Z of dimension $P \times P$ is the determinant of a submatrix of Z formed by removing k rows and the corresponding k columns of Z where $k \in \{0,\ldots,P-1\}$.

A strategy profile $s^* \in \mathcal{S}$ is called an equilibrium of $\Gamma^w(\bar{u})$ if and only if s^* solves each player's problem, i.e.,

$$s^* \in BR^w(s^*)$$

where $BR^w = (BR_1^w, \ldots, BR_P^w)$ is defined by, for all $p \in \mathcal{P}$, $s_{-p} \in \mathcal{S}_{-p}$,

$$\begin{aligned} &BR_p^w(s_{-p}) \\ &\quad := \operatorname*{argmin}_{s_p \in \mathcal{S}_p} c_p(s_p, s_{-p}) + w_p \left(\bar{u}_p - u_p(s_p, s_{-p})\right)^+ . \end{aligned}$$

We will denote the set of equilibria in the weighted cost minimization game $\Gamma^w(\bar{u})$ by $\mathcal{E}^w(\bar{u})$.

Our primary interest is in the case where w_p is large, for all $p \in \mathcal{P}$. When w_p is large, player p would be expected to achieve its target utility level, if at all possible, since otherwise player p would be penalized heavily. In this case, if a player p's target utility level is "too high" to achieve, then player p would incur a very "high cost". In a practical scenario, a player who cannot achieve its target utility level despite incurring a very high cost may be encouraged to downgrade its target utility level to a more "reasonable" level. However, regardless of the target utility levels $\bar{u} \in \mathbb{R}_+^P$, the weighted cost minimization game $\Gamma^w(\bar{u})$ always possesses equilibrium.

Proposition 7: For any $(\bar{u}, w) \in \mathbb{R}_+^P \times \mathbb{R}_+^P$, the weighted cost minimization game $\Gamma^w(\bar{u})$ possesses equilibrium.

Proof: Since $s_p = 0$ achieves the cost $w_p \bar{u}_p$, we have, for all $p \in \mathcal{P}$, $s_{-p} \in \mathcal{S}_{-p}$,

$$\hat{s}_p \in BR_p^w(s_{-p}) \quad \Rightarrow \quad c_p(\hat{s}_p, s_{-p}) \leq w_p \bar{u}_p.$$

Hence, BR^w maps $\mathcal{S}$ into the subsets of the set

$$\begin{aligned} \bar{\mathcal{S}}^w = \times_{p \in \mathcal{P}} \bar{\mathcal{S}}_p^w := \times_{p \in \mathcal{P}} \operatorname{cc}\left(\{s_p \in \mathcal{S}_p :\right. \\ \left. c_p(s_p, s_{-p}) \leq w_p \bar{u}_p,\ s_{-p} \in \mathcal{S}_{-p}\}\right). \quad (16) \end{aligned}$$

$\bar{\mathcal{S}}^w$ is nonempty, compact and convex; see Lemma 2. By Theorem 1, the restriction $\Gamma^w(\bar{u})|_{\bar{\mathcal{S}}^w}$ of $\Gamma^w(\bar{u})$ to $\bar{\mathcal{S}}^w$ possesses equilibrium; see Remark 1. Finally, any equilibrium of $\Gamma^w(\bar{u})|_{\bar{\mathcal{S}}^w}$ is also an equilibrium of $\Gamma^w(\bar{u})$. ∎

The following proposition states that any equilibrium of $\Gamma^w(\bar{u})$ achieving the target utility levels of all players must also be an equilibrium of $\Gamma^c(\bar{u})$. This result appeared in Theorem 1 in [15]; however, we state and prove it here for the sake of completeness.

Proposition 8: For any $(\bar{u}, w) \in \mathbb{R}_+^P \times \mathbb{R}_+^P$, we have

$$\{s \in \mathcal{E}^w(\bar{u}) : u_p(s) \geq \bar{u}_p, \text{ for all } p \in \mathcal{P}\} \subset \mathcal{E}^c(\bar{u})$$

Proof: Let $\hat{s} \in \mathcal{E}^w(\bar{u})$ be such that $u_p(\hat{s}) \geq \bar{u}_p$, for all $p \in \mathcal{P}$. We have, for all $p \in \mathcal{P}$, $s_p \in \mathcal{S}_p$,

$$\begin{aligned} c_p(\hat{s}) &= c_p(\hat{s}) + w_p \left(\bar{u}_p - u_p(\hat{s})\right)^+ \\ &\leq c_p(s_p, \hat{s}_{-p}) + w_p \left(\bar{u}_p - u_p(s_p, \hat{s}_{-p})\right)^+ . \end{aligned}$$

Hence, we have, for all $p \in \mathcal{P}$, $s_p \in \mathcal{S}_p$,

$$u_p(s_p, \hat{s}_{-p}) \geq \bar{u}_p \quad \Rightarrow \quad c_p(\hat{s}) \leq c_p(s_p, \hat{s}_{-p}).$$

∎

We now present a relationship between the equilibrium sets of $\Gamma^c(\bar{u})$ and $\Gamma^w(\bar{u})$, under an additional assumption.

Assumption 2: For all $p \in \mathcal{P}$, $s \in \mathcal{S}$, there exists some $\hat{h}_p(s) \in \mathcal{R}_{\mathcal{S}_p}(s_p)$ such that

$$u_p'(s_p, s_{-p}; \hat{h}_p(s)) > 0$$

and, for all $M > 0$,

$$\sup_{s \in \mathcal{S}: \|s\| \leq M} \frac{c_p'(s_p, s_{-p}; \hat{h}_p(s))}{u_p'(s_p, s_{-p}; \hat{h}_p(s))} < +\infty. \quad (17)$$

Remark 3: Assumption 2 holds, if, for all $p \in \mathcal{P}$, $s_{-p} \in \mathcal{S}_{-p}$, the functions $c_p(\cdot, s_{-p})$, $u_p(\cdot, s_{-p})$ are differentiable everywhere in $\mathcal{S}_p$ (which requires $c_p(\cdot, s_{-p})$, $u_p(\cdot, s_{-p})$ to be defined in an open set containing $\mathcal{S}_p$) and the gradients $\nabla_{s_p} c_p$, $\nabla_{s_p} u_p$ (with respect to s_p) are continuous in $\mathcal{S}$. To see this, for any $M > 0$, let $\hat{h}_p(s) := M s_p / \|s_p\|$, if $s_p \neq 0$; otherwise, let $\hat{h}_p(s) \in \mathcal{R}_{\mathcal{S}_p}(0)$ be arbitrary except $\left\|\hat{h}_p(s)\right\| = M$. It follows that

$$\begin{aligned} &\sup_{s \in \mathcal{S}: \|s\| \leq M} \frac{c_p'(s_p, s_{-p}; \hat{h}_p(s))}{u_p'(s_p, s_{-p}; \hat{h}_p(s))} \\ &\leq \sup_{s \in \mathcal{S}: \|s\| \leq M} \frac{c_p'(\hat{h}_p(s), s_{-p}; \hat{h}_p(s))}{u_p'(\hat{h}_p(s), s_{-p}; \hat{h}_p(s))} && (18) \\ &\leq \sup_{s \in \mathcal{S}: \|s_p\| = M, \|s_{-p}\| \leq M} \frac{\langle \nabla_{s_p} c_p(s_p, s_{-p}), s_p \rangle}{\langle \nabla_{s_p} u_p(s_p, s_{-p}), s_p \rangle} && (19) \\ &< +\infty. && (20) \end{aligned}$$

The inequality (18) follows from (i) for any $s_p \in \mathcal{S}_p$ satisfying $\|s_p\| \leq M$, there exists some $\hat{\alpha} \in [0, 1]$ such that $\hat{\alpha} \hat{h}_p(s) = s_p$, and (ii) $c_p'(\alpha \hat{h}_p(s), s_{-p}; \hat{h}_p(s)) / u_p'(\alpha \hat{h}_p(s), s_{-p}; \hat{h}_p(s))$ is nondecreasing with respect to $\alpha \in \mathbb{R}_+$, due to the convexity of $c_p(\cdot, s_{-p})$ and the concavity of $u_p(\cdot, s_{-p})$. The inequality (19) holds because, for all $s \in \mathcal{S}$ satisfying $\|s\| \leq M$, we have $\left\|\hat{h}_p(s)\right\| = M$ and $\|s_{-p}\| \leq M$. The inequality (20) follows from the continuity of $\nabla_{s_p} c_p$, $\nabla_{s_p} u_p$.

Proposition 9: Let Assumption 2 (as well as Assumption 1) hold. For any $\bar{u} \in \mathbb{R}_+^P$ and any bounded set $\mathcal{B} \in \mathcal{S}$, there exists some $\bar{w} \in \mathbb{R}_+^P$ such that, for all $w > \bar{w}$,

$$\mathcal{E}^c(\bar{u}) \cap \mathcal{B} \;=\; \mathcal{E}^w(\bar{u}) \cap \mathcal{B}.$$

Proof: Fix any $\bar{u} \in \mathbb{R}_+^P$ and any bounded set $\mathcal{B} \in \mathcal{S}$. Let $\bar{w} = (\bar{w}_1, \ldots, \bar{w}_P)$ be defined by, for all $p \in \mathcal{P}$,

$$\bar{w}_p := \sup_{s \in \mathcal{B}} \frac{c_p'(s_p, s_{-p}; \hat{h}_p(s))}{u_p'(s_p, s_{-p}; \hat{h}_p(s))} < +\infty$$

where $\hat{h}_p(\cdot)$ is as in Assumption 2. We first show that, for all $w \geq \bar{w}$, $\mathcal{E}^c(\bar{u}) \cap \mathcal{B} \subset \mathcal{E}^w(\bar{u}) \cap \mathcal{B}$. For this, assume that $\mathcal{E}^c(\bar{u}) \cap \mathcal{B} \neq \emptyset$. For each $s^* \in \mathcal{E}^c(\bar{u}) \cap \mathcal{B}$, there exist some Kuhn-Tucker vector $\left(\lambda_1(s_{-1}^*), \ldots, \lambda_P(s_{-P}^*)\right) \in \mathbb{R}_+^P$ (not necessarily unique[8]) such that, for all $p \in \mathcal{P}$,

$$c_p(s^*) = \min_{s_p \in \mathcal{S}_p} c_p(s_p, s_{-p}^*) + \lambda_p(s_{-p}^*) \left(\bar{u}_p - u_p(s_p, s_{-p}^*)\right).$$

[8]If the set $\Lambda_p(s_{-p}^*)$ of Kuhn-Tucker multipliers for player p's problem is not a singleton, then we take $\lambda_p(s_{-p}^*)$ as the smallest element in $\Lambda_p(s_{-p}^*)$, which exists because $\Lambda_p(s_{-p}^*)$ is a closed subset of $\mathbb{R}_+$

This results in, for all $s^* \in \mathcal{E}^c(\bar{u}) \cap \mathcal{B}$, $p \in \mathcal{P}$, $w_p \geq \lambda_p(s^*_{-p})$,

$$c_p(s^*) = \min_{s_p \in \mathcal{S}_p} c_p(s_p, s^*_{-p}) + w_p \left(\bar{u}_p - u_p(s_p, s^*_{-p})\right)^+.$$

Therefore, it follows that for all $s^* \in \mathcal{E}^c(\bar{u}) \cap \mathcal{B}$, $w \geq \left(\lambda_1(s^*_{-1}), \ldots, \lambda_P(s^*_{-P})\right)$, we have $s^* \in \mathcal{E}^w(\bar{u}) \cap \mathcal{B}$. In view of (7) and Assumption 2, we have, for all $p \in \mathcal{P}$,

$$\sup_{s^* \in \mathcal{E}^c(\bar{u}) \cap \mathcal{B}} \lambda_p(s^*_{-p}) \leq \bar{w}_p. \tag{21}$$

Therefore, we obtain, for all $w \geq \bar{w}$, $\mathcal{E}^c(\bar{u}) \cap \mathcal{B} \subset \mathcal{E}^w(\bar{u}) \cap \mathcal{B}$.

We now show that, for all $w > \bar{w}$, $\mathcal{E}^c(\bar{u}) \cap \mathcal{B} \supset \mathcal{E}^w(\bar{u}) \cap \mathcal{B}$. For this, assume that there exist $w^* \in \mathbb{R}^P_+$ and $s^* \in \mathcal{E}^{w^*}(\bar{u}) \cap \mathcal{B}$ such that, for some $p \in \mathcal{P}$, $u_p(s^*) < \bar{u}_p$. The optimality condition for $s^*_p \in BR^{w^*}_p(s^*_{-p})$ (similar to (7)) and Assumption 2 imply that

$$w^*_p \leq \frac{c'_p(s^*_p, s^*_{-p}; \hat{h}_p(s^*))}{u'_p(s^*_p, s^*_{-p}; \hat{h}_p(s^*))} \leq \bar{w}_p \tag{22}$$

where $\hat{h}_p(\cdot)$ is as in Assumption 2. Therefore, for all $w > \bar{w}$, $s \in \mathcal{E}^w(\bar{u}) \cap \mathcal{B}$, $p \in \mathcal{P}$, we have $u_p(s) \geq \bar{u}_p$. This implies that, for all $w > \bar{w}$, $\mathcal{E}^c(\bar{u}) \cap \mathcal{B} \supset \mathcal{E}^w(\bar{u}) \cap \mathcal{B}$. ∎

Remark 4: If $\bar{u} \notin \mathbb{U}_e$ (in other words $\mathcal{E}^c(\bar{u}) = \emptyset$), then any equilibrium in $\mathcal{E}^w(\bar{u})$ (which is nonempty for any $(\bar{u}, w) \in \mathbb{R}^P_+ \times \mathbb{R}^P_+$) will exit any given bounded set $\mathcal{B} \subset \mathcal{S}$ as $w \uparrow +\infty$. If $\mathcal{E}^c(\bar{u})$ is contained in some bounded set $\mathcal{B}$, then we have $\mathcal{E}^c(\bar{u}) = \mathcal{E}^w(\bar{u}) \cap \mathcal{B}$, for all sufficiently large $w \in \mathbb{R}^P_+$. If $\mathcal{E}^c(\bar{u})$ is unbounded, then arbitrarily large subsets of $\mathcal{E}^c(\bar{u})$ can be obtained through $\mathcal{E}^w(\bar{u})$ by choosing large enough $w \in \mathbb{R}^P_+$.

We provide an example where $\mathcal{E}^c(\bar{u})$ is unbounded.

Example 4: For all $p \in \mathcal{P} = \{1, 2\}$, let

$$\mathcal{S}_p = \mathbb{R}_+, \quad c_p(s_p, s_{-p}) = s_p, \quad u_p(s_p, s_{-p}) = \frac{s_p}{\max\{1, s_{-p}\}}$$

and $\bar{u} = (1, 1)$. It is straightforward to compute the set of equilibria in $\Gamma^c(\bar{u})$ as $\mathcal{E}^c(\bar{u}) = \{s \in \mathbb{R}^2_+ : 1 \leq s_1 = s_2\}$, which is an unbounded set. On the other hand, for any $w > (1, 1)$, we obtain $\mathcal{E}^w(\bar{u}) = \{s \in \mathbb{R}^2_+ : 1 \leq s_1 = s_2 \leq \min\{w_1, w_2\}\}$.

Remark 5: Proposition 9 is similar in spirit to Theorem 3 in [15] and Theorems 2.5 and 2.8 in [13]. However, the penalty updating algorithms in [15] and [13] rely on the gradient information, and hence it is required that the cost functions and the functions involved in the inequalities defining the set of feasible strategies are continuously differentiable (i.e., differentiable with continuous derivatives). Whereas, in this paper, penalty terms are simply chosen large enough to obtain the part of $\mathcal{E}^c(\bar{u})$ contained in any arbitrary set $\mathcal{B} \subset \mathcal{S}$ without any assumptions on continuous differentiability.

VII. Learning by Best Response Dynamics

We now consider an iterative learning process by which each player continually updates its strategy in response to the strategies of the other players in order to minimize its cost while achieving its target utility level. The learning process given below requires each player to compute a best response, at each step $t = 1, 2, \ldots$, to its competitors' strategies chosen in the previous step $t - 1$. Computing a best response itself requires each player to solve an optimization problem at each step, which we assume that each player can do.

Each player $p \in \mathcal{P}$ updates its strategy $s_p(t)$ according to

$$s_p(t+1) \in (1 - \alpha(t))s_p(t) + \alpha(t) BR^w_p(s_{-p}(t)) \tag{23}$$

where $\alpha(t) \in [0, 1]$ is player p's willingness to minimize its cost while achieving its target utility level. Choosing $\alpha(t) = 1$, for all $t \geq 1$, may cause players to be aggressive optimizers and hence may lead to slow convergence or oscillations. Decreasing $\alpha(t)$ in a gradual manner generally helps the players to learn their "optimal" strategies faster.

A close variant of the learning process (23) for each player $p \in \mathcal{P}$ is given by

$$s_p(t+1) \in (1 - \alpha(t))s_p(t) + \alpha(t) BR^c_p(s_{-p}(t)). \tag{24}$$

A subtlety that arises here is that the strategies produced by (24) may not satisfy the target utility levels. In other words, $u_p(s_p(t), s_{-p}(t)) \geq \bar{u}_p$ is not guaranteed; although $u_p(s_p(t), s_{-p}(t-1)) \geq \bar{u}_p$ is always guaranteed, by definition. However, if the process (24) converges to some $\hat{s} \in \mathcal{S}$, then $\hat{s}$ must achieve the target utility levels, because $\hat{s}$ must in fact be an equilibrium of the cost minimization game.

Our simulations of both processes (23)-(24) usually result in convergent behavior. Furthermore, a proof of convergence can be constructed if the cost and the utility functions are weakly coupled. Toward this end, we extend the notion of weak coupling introduced in Subsection V-B to the case where player strategies are (weakly) coupled through both their cost functions and their utility functions. More precisely, we consider $\eta-$parameterizations of the cost minimization and the utility maximization games such that, for $\eta \geq 0$, each player $p \in \mathcal{P}$ incurs the cost $c_p(s_p, \eta s_{-p})$ and receives the utility $u_p(s_p, \eta s_{-p})$ for the strategy profile $(s_p, s_{-p}) \in \mathcal{S}$. We refer to the cost minimization game corresponding to the coupling coefficient $\eta \geq 0$ and the target utility levels $\bar{u} \in \mathbb{R}^P_+$ as $\Gamma^c(\bar{u}, \eta)$. The next proposition shows that, when the coupling coefficient $\eta > 0$ is sufficiently small, the best response correspondence is a contraction, which leads to the desired convergence result; see Proposition 4.1 in [24].

Assumption 3 (for $\bar{u} \in \mathbb{R}^P_+$): For all $p \in \mathcal{P}$, $(s_p, s_{-p}) \in \mathcal{S}$,

(i) $\mathcal{C}_p(s_{-p}) = \{\check{s}_p \in \mathcal{S}_p : u_p(\check{s}_p, s_{-p}) \geq \bar{u}_p\} \neq \emptyset$

(ii) $c_p(\alpha s_p, s_{-p})$ and $u_p(\alpha s_p, s_{-p})$ are differentiable with respect to $\alpha \in \mathbb{R}_{++}$, i.e.,

$$c'_p(s_p, s_{-p}; s_p) = -c'_p(s_p, s_{-p}; -s_p)$$
$$u'_p(s_p, s_{-p}; s_p) = -u'_p(s_p, s_{-p}; -s_p)$$

(iii) c_p, u_p are locally Lipschitz in $\mathcal{S}$ and c'_p, u'_p are locally Lipschitz in $\mathcal{D}_p$

(iv) for any compact $\bar{\mathcal{S}} \subset \mathcal{S}$, there exists $\rho > 0$ such that, for all $(\bar{s}_p, \bar{s}_{-p}) \in \bar{\mathcal{S}}$, $h_p \in \mathcal{R}_{\mathcal{S}_p}(\bar{s}_p)$,

$$(u_p - c_p)'(\bar{s}_p, \bar{s}_{-p}; h_p) + (u_p - c_p)'(\bar{s}_p + h_p, \bar{s}_{-p}; -h_p) \geq \rho \|h_p\|^2$$

(v) $\alpha(t) \in [0, 1]$, for all $t \geq 1$, and $\sum_{k \geq 1} \alpha(k) = +\infty$.

Remark 6: Part (i) of Assumption 3 is to ensure that each player can achieve its target utility regardless of the strategies chosen by the other players, which is necessary for the best response correspondence to be nonempty-valued. Part (ii), (iii), and (iv) are technical assumptions that allow us to prove the contraction property for the best response correspondence. In particular, part (iv) strengthens the convexity of $c_p - u_p$ with respect to s_p. If c_p, u_p are twice continuously differentiable everywhere in $\mathcal{S}$ (which requires c_p, u_p to be defined in an open set containing $\mathcal{S}$) then part (ii) and (iii) would hold. Part (iv) would also hold if, in addition, the Hessian matrix $\nabla^2_{s_p,s_p}(c_p - u_p)$ (with respect to s_p) satisfies: for any compact $\bar{\mathcal{S}} \subset \mathcal{S}$, there exists $\rho > 0$ such that, for all $\bar{s} \in \bar{\mathcal{S}}$, $\nabla^2_{s_p,s_p}(c_p - u_p)(\bar{s}) - \rho I$ is positive semi-definite. Part (v) is used to show the convergence of the learning processes, and it would be satisfied if, for example, $\alpha(t) = 1/t$ or $\alpha(t) = \bar{\alpha} \in (0, 1]$, for $t \geq 1$.

Proposition 10: Consider the $\eta-$parameterization of $\Gamma^c(\bar{u})$ with the typical game $\Gamma^c(\bar{u}, \eta)$, where $\bar{u} \in \mathbb{R}^P_+$, $\eta \geq 0$. Let Assumption 3 (in addition to Assumption 1) hold. For any $w > \lambda(0)$ where $\lambda(0) = (\lambda_1(0), \cdots, \lambda_P(0))$ is some Kuhn-Tucker vector for the decoupled game, there exists $\bar{\eta} > 0$ such that, for all $\eta \in [0, \bar{\eta}]$, both recursions

$$s(t+1) = (1 - \alpha(t))s(t) + \alpha(t)BR^w(\eta s(t)), \quad s(1) \in \bar{\mathcal{S}}^w$$
$$s(t+1) = (1 - \alpha(t))s(t) + \alpha(t)BR^c(\eta s(t)), \quad s(1) \in \bar{\mathcal{S}}^w$$

for $t \geq 1$, converge to the unique equilibrium of $\Gamma^c(\bar{u}, \eta)$, where $\bar{\mathcal{S}}^w$ is as in (16).

Proof: We show in the Appendix that, for all sufficiently small $\eta > 0$, $BR^c(\eta\bar{\mathcal{S}}^w) = BR^w(\eta\bar{\mathcal{S}}^w) \subset \bar{\mathcal{S}}^w$ and the restrictions of $BR^c(\eta\,\cdot\,)$, $BR^w(\eta\,\cdot\,)$ to $\bar{\mathcal{S}}^w$ are single-valued contractions with some Lipschitz constant $L \in [0, 1)$. The contraction mapping theorem implies the existence of a unique equilibrium $s^* \in \bar{\mathcal{S}}^w$ such that $s^* = BR^c(\eta s^*) = BR^w(\eta s^*)$; see Theorem 1 on page 272 in [32]. If $\{s(t)\}_{t\geq 1}$, is generated by either recursion, then it is straightforward to see that, for all $t \geq 1$,

$$\|s(t+1) - s^*\| \leq \prod_{k=1}^{t} (1 - (1 - L)\alpha(k)) \|s(1) - s^*\|.$$

The desired result follows from

$$\lim_{t\geq 1} \prod_{k=1}^{t} (1 - (1 - L)\alpha(k)) = 0$$

which is due to part (v) of Assumption 3. ■

We finish this section by some examples where the learning process (24) does not converge.

Example 5: Consider the problem in Example 1 with the target utility levels $\bar{u} = (1, 1)$. Let $\tilde{s} := ((1, 1), (1, 1))$. For any $\rho \geq 0$, $BR^c(\rho\tilde{s}) = \gamma(1 + \rho\beta)\tilde{s}$, where $\gamma := \sqrt{2} - 1$. Therefore, if $\alpha(t) = 1$, for all $t \geq 1$, then the process (24) with the initial strategy $s(1) = 0 \in \mathbb{R}^2_+ \times \mathbb{R}^2_+$ generates

$$s(t) = \gamma\left(1 + \gamma\beta + \cdots + (\gamma\beta)^{t-2}\right)\tilde{s}, \quad t = 2, 3, \cdots$$

which is unbounded for $\beta \geq 1/\gamma = 1/(\sqrt{2} - 1)$. Recall that, for $\beta \geq 1$, $((1, 0), (0, 1))$ is an equilibrium of $\Gamma^u(\bar{c})$ where $\bar{c} = (1, 1)$, with the corresponding utility levels $(1, 1)$; see Example 2. Hence, for $\beta \geq 1$, despite $\bar{u} = (1, 1) \in \mathbb{U}_e$, the process (24) is nonconvergent.

Example 6: Let us reconsider the problem in Example 1 with the target utility levels $\bar{u} = (1, 1)$. Assume $\beta = 1$ so that $((1, 0), (0, 1))$ is an equilibrium of $\Gamma^c(\bar{u})$. For any $p \in \mathcal{P}$, $\theta \geq 0$, small $\epsilon > 0$, there exists some $\bar{\theta} > 0$ such that

$$BR^c_p((\theta + 1 - \epsilon, \theta)) = (\bar{\theta}, \bar{\theta} + 1 - \epsilon)$$
$$BR^c_p((\theta, \theta + 1 - \epsilon)) = (\bar{\theta} + 1 - \epsilon, \bar{\theta}).$$

Therefore, if $\alpha(t) = 1$, for all $t \geq 1$, then the process (24) with the initial strategy $s(1) = ((1, \epsilon), (0, 1 + \epsilon))$, for some small $\epsilon > 0$, generates the sequence

$$\begin{aligned} &((1,0),(\bar{\theta}_1, \bar{\theta}_1 + 1 - \epsilon)) \to ((\bar{\theta}_2 + 1 - \epsilon, \bar{\theta}_2), (0, 1)) \\ \to\ &((1,0),(\bar{\theta}_3, \bar{\theta}_3 + 1 - \epsilon)) \to ((\bar{\theta}_4 + 1 - \epsilon, \bar{\theta}_4), (0, 1)) \\ \to\ &((1,0),(\bar{\theta}_5, \bar{\theta}_5 + 1 - \epsilon)) \to \cdots \end{aligned}$$

for some $\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3, \bar{\theta}_4, \bar{\theta}_5 \cdots > 0$. The process is nonconvergent despite the initial strategy $s(1)$ satisfies the target utility levels, i.e., $u_p(s(1)) \geq \bar{u}_p = 1$, for all $p \in \mathcal{P}$, and $s(1)$ can be arbitrarily close to the equilibrium $((1, 0), (0, 1))$.

VIII. SIMULATION RESULTS

In this section, we numerically verify some of our theoretical results. We consider a cost minimization game arising from the problem of power minimization in a MIMO interference system (3)-(5) with the following data: $P = 3$, $H_{1,1} = H_{2,2} = H_{3,3} = 2I$, and

$$H_{1,2} = H_{2,1} = H_{3,1} = \frac{1}{2}\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix},$$
$$H_{1,3} = H_{2,3} = H_{3,2} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}$$

with two different target rate profiles $\bar{u}^1 = (4, 3, 2)$ and $\bar{u}^2 = (7.2, 5.4, 3.6)$. We simulate the best response dynamics (23) and (24) with $\alpha(t) = 1$, for all $t \geq 1$, some random initial conditions, and different values of w. We evaluate the best response functions BR^c and BR^w by using the water filling solutions given in our previous work; see Appendix I in [16]. The rate and power trajectories generated by (24) with $\bar{u}^1$ is shown in Figure 2. When this simulation is repeated with (23) using the same initial conditions and $w_p \geq 5\ln 2$, $p \in \{1, 2, 3\}$, the results are identical to those in Figure 2. Clearly, the results in Figure 2 are convergent and $\bar{u}^1$ is achieved at equilibrium. Fig 3 shows the rate and power trajectories generated by (24) with $\bar{u}^2$ where the dashed lines represent $\bar{u}^2$. Our repeated simulations of this case with random initial conditions generate trajectories similar to those in Fig 3 where the rate trajectories settle at values that are slightly lower than $\bar{u}^2$ but the power trajectories diverge. However, target rate profiles that are slightly smaller than $\bar{u}^2$ are achievable at equilibrium with bounded power trajectories (the results are not shown here). This suggest that $\bar{u}^2$ is outside but not too far from $\mathcal{U}_e$.

When we simulate (23) with $\bar{u}^2$ and increasing values of w, the results are always convergent in contrast to those in Fig 3. We only present the case where $w_1 = w_2 = w_3 = 20\ln 2$ in Figure 4 where the dashed lines again represent $\bar{u}^2$. As w gets larger, the rate trajectories settle at values that are closer

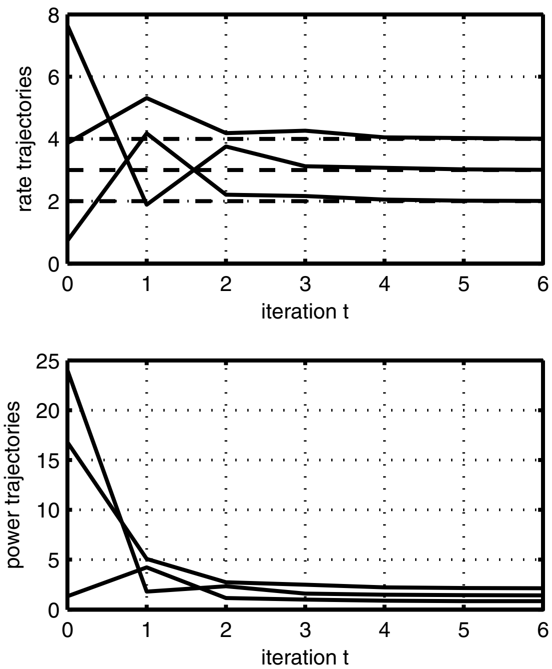

Fig. 2. The rate and power trajectories generated by (24) with $\bar{u}^1$.

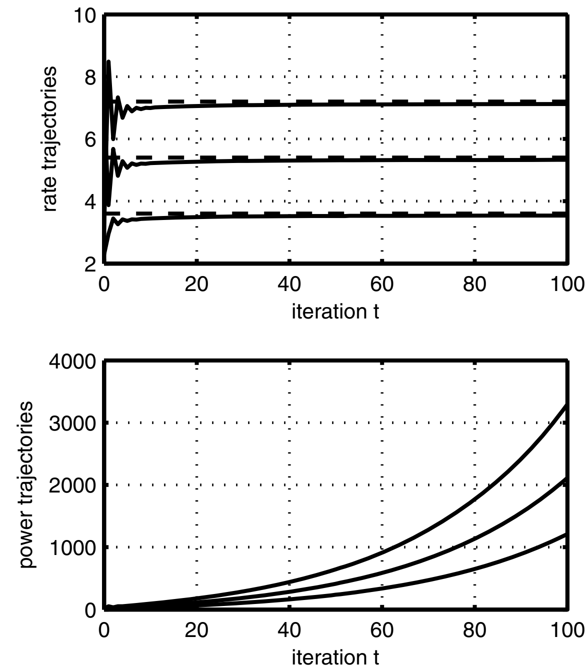

Fig. 3. The rate and power trajectories generated by (24) with $\bar{u}^2$.

to but still below $\bar{u}^2$ whereas the power trajectories settle at higher values, which is consistent with our theoretical results.

IX. CONCLUSIONS

This paper extends our previous work on power minimization in MIMO interference systems to general competitive resource allocation problems. Our extension involves a cost minimization and a utility maximization game, which are dual to each other. In the most generalized case, both the objective function and the strategy set of each player depend on the strategies of the other players. We obtain satisfactory counterparts of our previous results in our generalized setting by exploiting, in particular, a duality relation.

Obtaining less conservative estimates of the set of equilibrium utility levels is left for future work. Developing learning dynamics with a universally convergent behavior is another future research problem. Finally, improving the efficiency of equilibrium without requiring centralized coordination remains as yet another future research problem.

APPENDIX

If Assumption 3 (in addition to Assumption 1) holds, then the following statements are true.

1) BR^c is single-valued and locally Lipschitz in $\mathcal{S}$.
2) For any $w > \lambda(0)$ where $\lambda(0) = (\lambda_1(0), \ldots, \lambda_P(0))$ is some Kuhn-Tucker vector for the decoupled game, there exists $\bar{\eta} > 0$ such that, for all $\eta \in [0, \bar{\eta}]$,
   a) $BR^c(\eta \bar{\mathcal{S}}^w) = BR^w(\eta \bar{\mathcal{S}}^w) \subset \bar{\mathcal{S}}^w$
   b) The restrictions of $BR^c(\eta\ \cdot\ )$ and $BR^w(\eta\ \cdot\ )$ to $\bar{\mathcal{S}}^w$ are contractions

   where $\bar{\mathcal{S}}^w$ is the nonempty compact convex set defined in (16).

Proof: By parts (i) and (iv) of Assumption 3, BR^c and BR^w are single-valued in $\mathcal{S}$. Also, since $\bar{u}_p = 0 \Rightarrow BR^c_p = BR^w_p \equiv 0$, we consider below some $p \in \mathcal{P}$ for which $\bar{u}_p > 0$.

Part 1) will be proven in three steps.

Step 1: We show that BR^c_p can be locally restricted to compact subsets of $\mathcal{S}_p$. Fix any $\bar{s}_{-p} \in \mathcal{S}_{-p}$. Due to part (i) of Assumption 3, there exists $\bar{s}_p \in \mathcal{S}_p$ and $\bar{\epsilon} > 0$ such that, for all $s_{-p} \in \mathcal{B}_{-p} := \{\check{s}_{-p} \in \mathcal{S}_{-p} : \|\check{s}_{-p} - \bar{s}_{-p}\| \le \bar{\epsilon}\}$, we have $u_p(\bar{s}_p, s_{-p}) > \bar{u}_p$. Let $\bar{c}_p := \max_{s_{-p} \in \mathcal{B}_{-p}} c_p(\bar{s}_p, s_{-p}) < +\infty$. For all $s_{-p} \in \mathcal{B}_{-p}$, we must have $c_p(BR^c_p(s_{-p}), s_{-p}) \le \bar{c}_p$. Therefore,

$$BR^c_p(\mathcal{B}_{-p}) := \{BR^c_p(s_{-p}) : s_{-p} \in \mathcal{B}_{-p}\} \subset \hat{\mathcal{B}}_p$$

where $\hat{\mathcal{B}}_p := \mathrm{cc}\left(\{\check{s}_p \in \mathcal{S}_p : c_p(\check{s}_p, \check{s}_{-p}) \le \bar{c}_p, \check{s}_{-p} \in \mathcal{B}_{-p}\}\right)$. In view of Lemma 2, $\hat{\mathcal{B}}_p$ is nonempty, compact, and convex.

Step 2: We show that BR^c_p is continuous in $\mathcal{S}_{-p}$. Again, fix any $\bar{s}_{-p} \in \mathcal{S}_{-p}$, and let $\mathcal{B}_{-p}$, $\hat{\mathcal{B}}_p$ be as introduced above. Let $\widehat{BR}^c_p : \mathcal{B}_{-p} \rightrightarrows \hat{\mathcal{B}}_p$ be defined by

$$\widehat{BR}^c_p(s_{-p}) := \underset{s_p \in \hat{\mathcal{C}}_p(s_{-p})}{\operatorname{argmin}} \quad c_p(s_p, s_{-p})$$

where $\hat{\mathcal{C}}_p : \mathcal{B}_{-p} \rightrightarrows \hat{\mathcal{B}}_p$ is defined by

$$\hat{\mathcal{C}}_p(s_{-p}) := \{s_p \in \hat{\mathcal{B}}_p : u(s_p, s_{-p}) \ge \bar{u}_p\}.$$

For any $s_{-p} \in \mathcal{B}_{-p}$, we have $BR^c_p(s_{-p}) = \widehat{BR}^c_p(s_{-p})$. We show that $\widehat{BR}^c_p$ is continuous in $\mathcal{B}_{-p}$. By assumption, c_p is

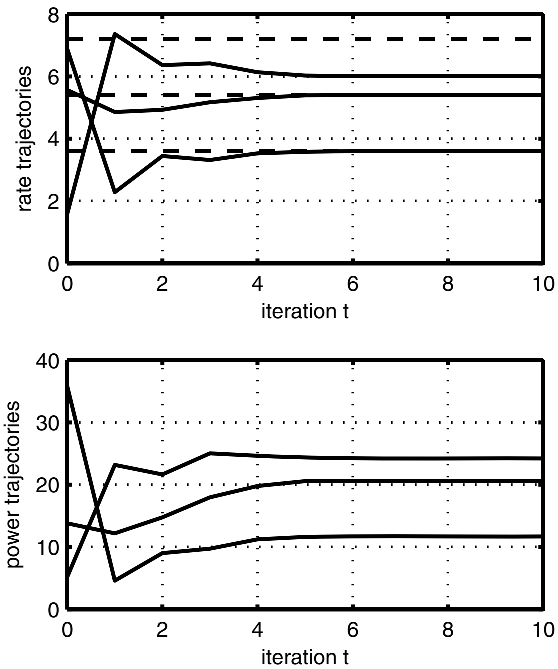

Fig. 4. The rate and power trajectories generated by (23) with $\vec{w}^2$.

continuous in $\mathcal{B}_{-p} \times \hat{\mathcal{B}}_p$. Also, $\hat{\mathcal{C}}_p$ is nonempty-valued, and compact-valued in $\mathcal{B}_{-p}$. Furthermore, $\hat{\mathcal{C}}_p$ is both u.s.c. and l.s.c. in $\mathcal{B}_{-p}$; see Remark 2. The Maximum Theorem implies that $\widehat{BR}_p^c$ is u.s.c. in $\mathcal{B}_{-p}$; see Theorem 2.3.1 in [29]. Since $\widehat{BR}_p^c$ is single-valued, it is continuous.

Step 3: We show that BR_p^c is locally Lipschitz in $\mathcal{S}_{-p}$ by following along the similar lines of the proof of Theorem 4.51 in [33]. For any $s_{-p} \in \mathcal{S}_{-p}$, player p's problem has a unique solution $\hat{s}_p = BR_p(s_{-p})$ satisfying (7). From (7) and part (ii) of Assumption 3, we have, for any $s_{-p} \in \mathcal{S}_{-p}$,

$$c'_p(\hat{s}_p, s_{-p}; \hat{s}_p) - \lambda_p(s_{-p}) u'_p(\hat{s}_p, s_{-p}; \hat{s}_p) = 0$$

where $\hat{s}_p = BR_p(s_{-p}) \neq 0$. Since $\hat{s}_p \neq 0$ implies $u'_p(\hat{s}_p, s_{-p}; \hat{s}_p) > 0$, $\lambda_p(s_{-p})$ is unique and given by

$$\lambda_p(s_{-p}) = \frac{c'_p(\hat{s}_p, s_{-p}; \hat{s}_p)}{u'_p(\hat{s}_p, s_{-p}; \hat{s}_p)} > 0. \tag{25}$$

Fix any compact $\bar{\mathcal{S}}_{-p} \subset \mathcal{S}_{-p}$. Let $\bar{\mathcal{S}}_p := BR_p^c(\bar{\mathcal{S}}_{-p}) = \{BR_p^c(s_{-p}) : s_{-p} \in \bar{\mathcal{S}}_{-p}\}$. Since BR_p^c is continuous in $\mathcal{S}_{-p}$, $\bar{\mathcal{S}}_p$ is compact. Also, since $0 \notin \bar{\mathcal{S}}_p$, there exists some $\epsilon' > 0$ such that, for all $(s_p, s_{-p}) \in \bar{\mathcal{S}}_p \times \bar{\mathcal{S}}_{-p}$, $u'(s_p, s_{-p}; s_p) \geq \epsilon'$. Therefore, since BR_p^c is continuous in $\mathcal{S}_{-p}$ and c'_p, u'_p are assumed to be locally Lipschitz in $\mathcal{D}_p$, λ_p is continuous in $\mathcal{S}_{-p}$. Moreover, the function c'_p/u'_p is Lipschitz in

$$\{(s_p, s_{-p}, h_p) \in \bar{\mathcal{S}}_p \times \bar{\mathcal{S}}_{-p} \times \bar{\mathcal{S}}_p : s_p = h_p\}.$$

Thus, there exist some $L_1 \geq 0$ such that, for all $s^1_{-p}, s^2_{-p} \in \bar{\mathcal{S}}_{-p}$,

$$\left|\lambda_p(s^1_{-p}) - \lambda_p(s^2_{-p})\right| \leq L_1 \left(\|\Delta_p\| + \|\Delta_{-p}\|\right) \tag{26}$$

where $s^i_p := BR^c_p(s^i_{-p})$, $i \in \{1,2\}$, $\Delta_p := s^2_p - s^1_p$, $\Delta_{-p} := s^2_{-p} - s^1_{-p}$. For all $s^1_{-p}, s^2_{-p} \in \bar{\mathcal{S}}_{-p}$, let

$$\begin{aligned}
&\Delta(s^1_{-p}, s^2_{-p}) := \\
&- \left(c'_p(s^1_p, s^2_{-p}; \Delta_p) - \lambda_p(s^1_{-p}) u'_p(s^1_p, s^2_{-p}; \Delta_p)\right) \\
&- \left(c'_p(s^2_p, s^2_{-p}; -\Delta_p) - \lambda_p(s^2_{-p}) u'_p(s^2_p, s^2_{-p}; -\Delta_p)\right) \qquad (27) \\
&+ \left(c'_p(s^1_p, s^1_{-p}; \Delta_p) - \lambda_p(s^1_{-p}) u'_p(s^1_p, s^1_{-p}; \Delta_p)\right) \\
&- \left(c'_p(s^1_p, s^1_{-p}; \Delta_p) - \lambda_p(s^1_{-p}) u'_p(s^1_p, s^1_{-p}; \Delta_p)\right) \qquad (28) \\
&+ \left(\lambda_p\left(s^1_{-p}\right) - \lambda_p\left(s^2_{-p}\right)\right)\left(u_p\left(s^1_p, s^2_{-p}\right) - u_p\left(s^1_p, s^1_{-p}\right)\right)
\end{aligned}$$

where s^1_p, s^2_p, Δ_p, Δ_{-p} are as introduced above. Note that $\Delta_p \in \mathcal{R}_{\mathcal{S}_p}(s^1_p)$, $-\Delta_p \in \mathcal{R}_{\mathcal{S}_p}(s^2_p)$), and the terms in (27), (28) are nonpositive. Also, λ_p is uniformly bounded in $\bar{\mathcal{S}}_{-p}$, and it satisfies (26). In addition, for all $(s_p, s_{-p}) \in \mathcal{S}$, $c'_p(s_p, s_{-p}; \,\cdot\,)$, $u'_p(s_p, s_{-p}; \,\cdot\,)$ are positively homogenous in $\mathcal{R}_{\mathcal{S}_p}(s_p)$. Hence, by part (iii) of Assumption 3, there exists an $L_2 \geq 0$ such that, for all $s^1_{-p}, s^2_{-p} \in \bar{\mathcal{S}}_{-p}$,

$$\Delta(s^1_{-p}, s^2_{-p}) \leq L_2 \left(\|\Delta_p\| + \|\Delta_{-p}\|\right) \|\Delta_{-p}\|.$$

On the other hand, we have, for all $s^1_{-p}, s^2_{-p} \in \bar{\mathcal{S}}_{-p}$,

$$\begin{aligned}
&\Delta(s^1_{-p}, s^2_{-p}) = \\
&- \left(c'_p(s^1_p, s^2_{-p}; \Delta_p) - \lambda_p(s^1_{-p}) u'_p(s^1_p, s^2_{-p}; \Delta_p)\right) \\
&- \left(c'_p(s^2_p, s^2_{-p}; -\Delta_p) - \lambda_p(s^2_{-p}) u'_p(s^2_p, s^2_{-p}; -\Delta_p)\right) \\
&+ \left(\lambda_p\left(s^1_{-p}\right) - \lambda_p\left(s^2_{-p}\right)\right)\left(u_p\left(s^1_p, s^2_{-p}\right) - u_p\left(s^2_p, s^2_{-p}\right)\right). \\
&\geq -c'_p(s^1_p, s^2_{-p}; \Delta_p) - c'_p(s^2_p, s^2_{-p}; -\Delta_p) \\
&\quad + \min\left\{\lambda_p(s^1_{-p}), \lambda_p(s^2_{-p})\right\} \\
&\qquad \left(u'_p(s^1_p, s^2_{-p}; \Delta_p) + u'_p(s^2_p, s^2_{-p}; -\Delta_p)\right)
\end{aligned}$$

where the inequality follows from the concavity of $u_p(\,\cdot\,, s_{-p})$, for any $s_{-p} \in \mathcal{S}_{-p}$. Due to part (iv) of Assumption 3, we have

$$L_3 \|\Delta_p\|^2 \leq \Delta \leq L_2 \left(\|\Delta_p\| + \|\Delta_{-p}\|\right) \|\Delta_{-p}\|$$

where $L_3 := \rho \min\left\{1, \min_{s_{-p} \in \bar{\mathcal{S}}_{-p}} \lambda_p(s_{-p})\right\} > 0$ and $\rho > 0$ is as in part (iv) of Assumption 3 for $\bar{\mathcal{S}} = \bar{\mathcal{S}}_p \times \bar{\mathcal{S}}_{-p}$. It follows that

$$\|\Delta_p\| \leq \frac{L_2 + \sqrt{L_2^2 + 4 L_2 L_3}}{2 L_3} \|\Delta_{-p}\|. \tag{29}$$

To prove part 2), recall that λ_p is continuous in $\mathcal{S}_{-p}$. Hence, for any $w_p > \lambda_p(0)$, there exists some $\bar{\eta}_p > 0$ such that, for all $\eta \in [0, \bar{\eta}_p]$,

$$w_p \geq \max_{s_{-p} \in \bar{\mathcal{S}}^w_{-p}} \lambda_p(\eta s_{-p}).$$

Thus, for any $w > \lambda(0)$, there exists some $\bar{\eta} > 0$ such that, for all $\eta \in [0, \bar{\eta}]$, we have $BR^c(\eta \bar{\mathcal{S}}^w) = BR^w(\eta \bar{\mathcal{S}}^w)$. Since BR^w maps $\mathcal{S}$ into the subsets of the $\bar{\mathcal{S}}^w$ (see the proof of part 1) in Proposition 7), we have $BR^c(\eta \bar{\mathcal{S}}^w) = BR^w(\eta \bar{\mathcal{S}}^w) \subset \bar{\mathcal{S}}^w$.

Finally, since BR^c is locally Lipschitz in $\mathcal{S}$, there exists $L_{\bar{\eta}} \geq 0$, such that, for all $s^1, s^2 \in \bar{\mathcal{S}}^w$,

$$\left\|BR^c(\bar{\eta} s^1) - BR^c(\bar{\eta} s^2)\right\| \leq L_{\bar{\eta}} \left\|s^1 - s^2\right\|.$$

Hence, for all $L \in [0, 1)$, $\eta \in [0, L\bar{\eta}/L_{\bar{\eta}}]$, $s^1, s^2 \in \bar{\mathcal{S}}^w$,

$$\left\|BR^c(\eta s^1) - BR^c(\eta s^2)\right\| \leq L \left\|s^1 - s^2\right\|.$$

■

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