Constrained State Estimation and Control over Communication Networks

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***Abstract*— This paper performs a joint analysis of control and coding for stability and performance of LTI systems connected over communication networks. We study the communication rate required for stability of the differential entropy and mean-square stability of the state estimation error. We show that the optimal control and coding problems in the minimization of an LQR cost are separable; we further show that the optimal control is linear in its argument and we provide the solution to the optimal quantization problem. Mean-square stability of an LTI system, as a function of the rate and network reliability, is also studied.**

***Keywords:* Quantization, Communication rate, Communication constrained control, Optimal control**

I. Introduction and problem definition

Classical control analysis assumes perfect information sharing between the system plant and the decision maker/controller, with no delay, no precision effects, and no communication constraints. However, in recent applications such as control over wireless systems, control over the Internet, control in decentralized and distributed systems, and digital control of automatic systems, a communication-theoretic approach as well as a non-traditional control approach is crucial to the design of a system with optimal performance under a specified criterion. Communication networks might be specifically important in applications such as the control of a plant by a supercomputer which can handle complex tasks which the plant cannot, or remote control where the controller/decision maker uses Internet to control a robot. Along these general lines, some of the related papers are [5], [6], [4], [7], [11], [3], [9], [10].

The basic model we introduce here focuses on the unreliable character of the links in both directions (see Fig. 1). We model this character of the links by a Bernoulli process, in which links fail, or equivalently packets are lost, independently. Thus, the controller has access to the measured output only at times governed by the underlying link-failure, or packet-loss, process. Similarly, the plant has access to the controller output only at times governed by an independent second link-failure process. Losses introduce an uncertainty in the state to be controlled, and the system is to be designed so as to achieve a degree of stability in state estimation error as well as in the state itself. For the state estimation problem, stability in mean-square estimation error and in differential entropy of the estimation error is considered. With the information structure induced by encoding and uncertainties in the network, optimal control is determined by minimizing some performance index.

The problem of control of a linear system over a network with packet losses can be modeled as a Markov jump linear system if the controller has an appropriate information set, and then the standard solutions are directly applicable [14], [15], [16]. However, for communicating over a network, the information structure of the controller does not permit a direct application of the theory of jump linear systems to obtain the solution. Thus, we resort here to basic dynamic-programming to derive the optimal controllers that minimize the quadratic cost under the induced information structure. We show that separation of estimation and control holds, thus the optimal controller can be designed independently from the optimal coder, which is also a solution of a separate dynamic programming equation. We finally analyze conditions on mean-square stability of an LTI system.

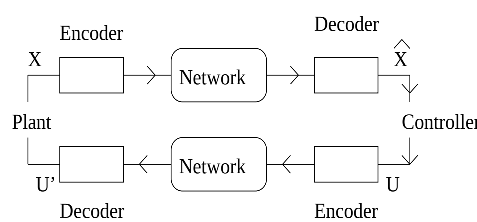

Fig. 1. Control over networks.

In a control setting, the failures in the transmission of the control signal potentially leads to a *dual effect* of the control values in the state estimation error, for if the controller does not know whether its message was successfully received or not the estimation error would be a function of the control signals as well. In this case the expected estimation error would have a linear contribution from the control signal [12]. However, for the popular communication protocol currently used in the Internet, TCP/IP (Transmission Control Protocol), for

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instance, feedback is rezlized by ACK (acknowledgement) signals. For the case with UDP (User Datagram Protocol), on the other hand where there is no explicit feedback within the protocol, one could embed an ACK signal in the data sent and received, and can eliminate the dual effect of the control [1]. We finally note that, optimal control without explicit feedback is analyzed in [12] and [10].

For future reference, we write the state equation for LTI systems as

$$x_{t+1} = Ax_t + \alpha_t B u_t \quad (1)$$

$$z_t = \beta_t f(x_t) \quad (2)$$

where x_t is the state vector, u_t is the control, $f(.)$ is the causal encoding vector which has access to all the past encoded and actual values as well as the sequence β_0^{t-1}, and finally z_t is the observation vector (all vectors are of appropriate dimensions). The initial state x_0 is also random and is characterized by a given probability distribution P_{x_0} which is continuous over a compact support set. The plant is remotely controlled over a network with links that are prone to failure. Link failures correspond to packet losses. The stochastic process α_t models this unreliable nature of the links. Basically, $\alpha_t = 0$ (zero control) when the link from the controller to the plant (uplink) fails, i.e. the control packet is lost, and $\alpha_t = 1$, otherwise. We will assume the former to take place with probability α. Likewise the observation is also subject to failures and the controller receives information only if $\beta_t = 1$ which is assumed to take place with probability $1 - \beta$. The plant is assumed to be incapable of making intelligent decisions when the uplink fails, and the control data is lost. However, an alternative strategy for the plant that does not require much intelligence could be to use the last available control instead of using zero control when the link fails. This case can be captured with the above model if one defines the control u_t as a state $\bar{x}_t$, adds this new state to the existing state vector to obtain the augmented state $[x_t \ \bar{x}_t]^T$, and rewrites the state equation by using the fact that $\bar{x}_t$ will be u_t if the link is up, and it will be $\bar{x}_{t-1}$ if it is down. Since the last available control case can be converted into the zero control one by augmenting the state space, for most of the discussions we only consider the zero control case.

We start in section II with the problem of state estimation through communication networks. We then study the optimal control and encoding of an LTI system over a network in section III, where the structure of the optimal control and the conditions for stability in the mean-square are presented. Section IV includes concluding remarks.

II. State estimation over a communication network

In this section, we analyze the problem of state tracking, where we use differential entropy of the error and the mean-square error as measures of uncertainty in the information the controller has with regard to the state at the plant.

Let $\hat{x}_t$ be the estimate of state at the controller and $e_t = x_t - \hat{x}_t$ be the state estimation error. We now have a few definitions.

Definition 2.1: An error system is stable in the mean-square sense if the second moment of the error, $E[||e_t||^2]$, converges to zero.

Definition 2.2: An error system is stable in the differential entropy of the estimation error if $\limsup_{t\to\infty} E(H(e_t|\hat{x}_t)) = -\infty$.

Definition 2.3: Let $\{\delta_t^i, \quad i = 1, 2, \ldots, K+1\}$, and $\{q_t^i, \quad i = 1, 2, \ldots K\}$ be the bin edges and the reconstruction values, respectively, at time t of a K-level scalar quantizer. Let δ_t and q_t denote the $(K+1)-$ and $K-$dimensional vectors corresponding to these sets. Then, such a quantizer is *time invariant* if the vectors δ_t and q_t change over time only up to a scaling constant, that is for some constant α and for all i and t,

$$\delta_{t+1}^i = \alpha \delta_t^i, \quad q_{t+1}^i = \alpha q_t^i \quad (3)$$

A. Mean-Square Stability in State Estimation

1) Convergence to the uniform density:

Lemma 2.4: Let x_0 be the realization of a scalar random variable X_0 with a continuous pdf $f_0(.)$ and with finite support $[-\Delta_0/2, \Delta_0/2]$. Suppose that at each time step, $x_{t+1} = ax_t$ is quantized successively using a K level uniform quantizer with a bin size equal to $\Delta_0|a|^t/K^{s(t)+1}$, for each time t, where $s(t) \leq t$ is the number of quantization operations performed until time t. If $\lim_{t\to\infty} \frac{s(t)}{t} = \eta \in (0, 1]$ then, the Kullback-Leibler distance between the zero-mean uniform error density with bin size $\Delta_n = \Delta_0|a^t|/K^{s(t)}$ and the quantization error density (conditioned on the sample path) converges to zero as $t \to \infty$ [1].

Remark Assuming that a stationary binomial distribution exists for the packet losses through the network from the state to the controller, and the loss probability is β, then η will be equal to $1 - \beta$ in Lemma 2.4. ⋄

Henceforth, in this subsection, we investigate the conditions on the communication rate to ensure stability in mean-square state estimation error.

We now assume that A is diagonalizable with possibly complex eigenvalues, and that the initial state x_0 is an outcome of a random variable with a bounded distribution. Since the control values do not have dual effects, they do not contribute to the state estimation error, thus the system we have is effectivelya control free system, $x_{t+1} = Ax_t$.

Proposition 2.5: Consider the LTI system in (1), connected through a communication network, with packet loss probability β. Let the eigenvalues of the system matrix A are $\lambda_i, \quad i = 1, \ldots, n$. Suppose the system is diagonalized and a uniform quantizer with K_i number of levels (bins) is used for each of the components of the state vector. Then the following bound has to be satisfied for each component of the vector to achieve mean-square stability in state estimation:

$$K_i > \sqrt{(1-\beta)/(\frac{1}{|\lambda_i|^2} - \beta)}. \tag{4}$$

Proof. For each of the components of the state vector, let e_t^i denote the support size of the state estimation error of the ith component at time t.

Now, asymptotically we have the quantization error as uniformly distributed. First let us assume that the eigenvalues are real. In case there is a packet loss, the expected mean-square state estimation error will be scaled by $|\lambda_i|^2$:

$$E[(e_{t+1}^i)^2] = |\lambda_i|^2 E[(e_t^i)^2].$$

On the other hand, if there is a successful transmission, the expected mean-square state estimation error will be

$$E[(e_{t+1}^i)^2] = (|\lambda_i|/K_i)^2 E[(e_t^i)^2],$$

where K_i is the number of levels in the quantizer of the ith component which is assumed to be uniform.

Thus the value of the second moment at time t, $E[(e_t^i)^2]$, will be

$$\begin{aligned} \beta|\lambda_i|^2 E[(e_{t-1}^i)^2] + (1-\beta)(|\lambda_i|/K)^2 E[(e_{t-1}^i)^2] \\ = |\lambda_i|^2(\beta + (1-\beta)/K^2) E[(e_{t-1}^i)^2]. \end{aligned} \tag{5}$$

To achieve stability, $|\lambda_i|^2(\beta + (1-\beta)/K_i^2)$ has to be less than unity in magnitude. Thus, K_i has to satisfy

$$K_i > \sqrt{(1-\beta)/(\frac{1}{|\lambda_i|^2} - \beta)}. \tag{6}$$

In case an eigenvalue is complex, then there will be a shift in the orientation of the quantizer. However the same updates will apply in the bin edge and reconstruction values and what matters will be the the magnitude of the eigenvalue. ⋄

Remark With fixed-length coding (since the density is asymptotically uniform) the minimum rate required would be

$$R = \sum_i \log_2(\lceil \sqrt{(1-\beta)/(1/|\lambda_i|^2 - \beta)} \rceil).$$

With variable-rate time-invariant coding one could further decrease the rate (minimizing the effect of the integer constraint on the number of levels), and get closer to the value in (6) [2]. ⋄

Remark Note that if $K = 2$, the condition in (4), for λ_i, will become:

$$|\lambda_i| < \frac{2}{\sqrt{1+3\beta}.}$$

Although the approach taken here is different, the result is in agreement with [4]. ⋄

B. Stability in differential entropy of the estimation error

Definition 2.6: Let there be a random variable X at a source, and another one Y at the receiver. Information rate of a code, Z, transmitted over the channel to improve the estimation at the receiver is

$$R_{X,Y}(Z) = I(X;Y) - I(X;Y|Z).$$

Define the average information rate as:

$$R_{avg,T} = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} R_t.$$

We now have the following proposition:

Proposition 2.7: Consider the system (1). For stability in the differential entropy of the state estimation error in this system connected through a communication network with packet loss probability of β, the average information rate required is at least $\max(0, \frac{\log_2(|A|)}{1-\beta})$, where $|A|$ is the determinant of A.

Proof. If the system is stable, then there is no need for communication, thus we will consider the case where the system is unstable. Now the information rate of the code $\hat{x}_t$ at time t is given by,

$$R_t = I(\hat{x}_t; x_t|\hat{x}_0^{t-1}) - I(\hat{x}_t; x_t|\hat{x}_0^t) \tag{7}$$

where $\hat{x}_0^{t-1}$ denotes the past information on the quantizer outputs, which are available both at the transmitter and the receiver (thus the ones which are successfully received by the controller). Since the system under consideration is Markov, the entire information about the state given all the past is captured in the previous output. The optimal encoder (in the sense of minimization of any measurable objective) would only use the last outcome and the receiver's state, $\hat{x}$ (see [8]). Thus, for a given arbitrary code, the information rate, R_t, is Hence,

$$H(x_t - \hat{x}_t|\hat{x}_t) = H(x_t - \hat{x}_t|\hat{x}_{t-1}) - R_t. \tag{8}$$

Since $x_t = Ax_{t-1}$, and $H(AX) = \log_2(|A|) + H(X)$ for any random variable X and matrix A, the above yields:

$$H(x_t - \hat{x}_t|\hat{x}_t) = \log_2(|A|) + H(x_{t-1} - \hat{x}_{t-1}|\hat{x}_{t-1}) - R_t\,. \tag{9}$$

However, if there is a packet loss, then the packets carrying data will not have an impact on the evolution of the entropy. In this case

$$H(x_t - \hat{x}_t|\hat{x}_t) = \log_2(|A|) + H(x_{t-1} - \hat{x}_{t-1}|\hat{x}_{t-1}). \tag{10}$$

Thus, once could regard the entropy as a state with a stochastic evolution, following (9) with probability $1-\beta$ and following (10) with probability β. Since the packet failures have a stationary distribution, we have

$$E[H(x_t - \hat{x}_t|\hat{x}_t)] = t\log_2(|A|) - (1-\beta)\sum_{t=0}^{T-1} R_t + H(x_0)$$

Thus, for an arbitrary $\epsilon > 0$

$$\limsup_T \frac{1}{T}[\sum_{t=1}^{T-1}[\log_2(|A|) - (1-\beta)R_t] + H(x_0)] \le -\epsilon \quad (11)$$

To achieve this, we need to have;

$$R_{avg,t} = \frac{1}{T}\sum_{t=0}^{T-1} R_t > \frac{1}{1-\beta}\log_2(|A|) + \epsilon. \quad (12)$$

$\diamond$

Remark Entropy is not an appropriate measure for multi-dimensional control systems. Suppose A has two eigenvalues, 2 and $1/2$. In this case, the determinant is 1, and the uncertainty measured in differential entropy will not increase, although clearly asymptotically the estimation error is unstable in one direction. Furthermore, as pointed out in [4], classical Shannon theory is not appropriate for such control systems even for scalar systems. Nonetheless, $\frac{1}{1-\beta}\log_2(|A|)$ is a lower bound on the rate. Achieving this information bound might not always be possible in a practical situation. The bound would be tight for instance, for a scalar system, since uniform density is the stationary density (up to a support set variation) where the same encoding scheme could be used. At least for the case where $\beta = 0$ and the system is scalar, entropy is a useful measure [2] for the characterization of optimal quantization. $\diamond$

Remark: In the internet, usually packet errors take place in bursts, thus one can improve the rate requirements by sending less data when the network is likely to be congested. A Markov model can be used to capture the state of reliability of the network. Using the Markov model, let β_r and β_u be the steady state probabilities of the reliable mode and the unreliable mode, respectively, and $\beta(u|r), \beta(r|u)$ denote the excursion probabilities. The rate can be further reduced by sending no data, except header bits to track the channel status. However the net information rate required will not change. Since typically the probability of excursion from the reliable state to the unreliable mode is much lower than the steady state probability of the unreliable mode, the data rate sent might decrease significantly. $\diamond$

III. Control of an LTI system over a network

A. Optimal LQR control, quantizer over a communication network

Consider the linear plant dynamics along with the measurement equation (1,2). The information available to the controller at time t is:

$$I_t = (z_0, z_1, \ldots, z_t; u_0, u_1, \ldots, u_{t-1}; \beta_0, \beta_1, \ldots, \beta_t; \alpha_0, \alpha_1, \ldots, \alpha_{t-1}), \ t = 1, 2, \ldots, T-1$$

The objective is to minimize the expected cost

$$J_\pi = E_{x_0, \alpha_t, \beta_t, t\in[0,T-1]}\{x_T^T Q_T x_T + \sum_{t=0}^{T-1} x_t^T Q_t x_t + u_t^T R_t u_t\} \quad (13)$$

over $\pi = \{\mu_0(I_0), \ldots, \mu_t(I_t)\}$ where I_t was defined above. From the dynamic-programming equation we have

$$\begin{aligned} J_{T-1}(I_{T-1}) &= \min_{u_{T-1}}[E\{x_{T-1}^T Q_{T-1} x_{T-1} \\ &+ u_{T-1}^T R_{T-1} u_{T-1} + (Ax_{T-1} + \alpha_{T-1} B \\ &\cdot u_{T-1})^T Q_T (Ax_{T-1} + \alpha_{T-1} B u_{T-1})|I_{T-1}, u_{T-1}\}] \\ &= E\{x_{T-1}^T (A^T Q_T A + Q_{T-1}) x_{T-1}|I_{T-1}\} \\ &+ \min_{u_{T-1}}[u_{T-1}^T (R_{T-1} + (1-\alpha) B^T Q_T B) u_{T-1} \\ &+ 2(1-\alpha) E\{x_{T-1}|I_{T-1}\}^T A^T Q_T B u_{T-1}] \end{aligned}$$

The minimization yields the optimal policy for the last stage:

$$\begin{aligned} u_{T-1} &= \mu_{T-1}^*(I_{T-1}) \\ &= -(1-\alpha)(R_{T-1} + (1-\alpha)B^T Q_T B)^{-1} \\ &\quad B^T Q_T A E\{x_{T-1}|I_{T-1}\} \end{aligned} \quad (14)$$

Upon substitution, we obtain

$$\begin{aligned} J_{T-1}(I_{T-1}) &= E\{x_{T-1}^T K_{T-1} x_{T-1}|I_{T-1}\} \\ &+ E\{(x_{T-1} - E\{x_{T-1}|I_{T-1}\})^T \\ &\cdot P_{T-1}(x_{T-1} - E\{x_{T-1}|I_{T-1}\})|I_{T-1}\} \end{aligned}$$

where the matrices K_{T-1} and P_{T-1} are given by

$$\begin{aligned} P_{T-1} &= (1-\alpha)^2 A^T Q_T B (R_{T-1} \\ &+ (1-\alpha) B^T Q_T B)^{-1} B^T Q_T A \end{aligned}$$

$$K_{T-1} = A^T Q_T A - P_{T-1} + Q_{T-1}$$

Note that the optimal policy (14) is a linear function of the conditional expectation $E\{x_{T-1}|I_{T-1}\}$. Now the DP equation for period $T-2$ is

$$\begin{aligned} J_{T-2}(I_{T-2}) &= \min_{u_{T-2}}[E\{x_{T-2}^T Q_{T-2} x_{T-2} \\ &+ u_{T-2}^T R_{T-2} u_{T-2} + J_{T-1}(I_{T-1})|I_{T-2}, u_{T-2}\}] \\ &= E\{x_{T-2}^T Q_{T-2} x_{T-2}|I_{T-2}\} + \min_{u_{T-2}}[u_{T-2}^T R_{T-2} u_{T-2} \\ &+ E\{x_{T-1}^T K_{T-1} x_{T-1}|I_{T-2}, u_{T-2}\}] \\ &+ E\{(x_{T-1} - E\{x_{T-1}|I_{T-1}\})^T P_{T-1} \\ &\cdot (x_{T-1} - E\{x_{T-1}|I_{T-1}\})|I_{T-2}, u_{T-2}\} \end{aligned}$$

One can exclude the last term from the minimization with respect to u_{T-2}, as there is no dual effect of the control, i.e. $x_t - E\{x_t|I_t\}$ is not a function of the control signals due to the acknowledgments. Minimization yields

$$\begin{aligned} u_{T-2} &= \mu^*_{T-2}(I_{T-2}) \\ = -(1-\alpha)(R_{T-2} &+ (1-\alpha)B^T K_{T-1} B)^{-1} \\ & B^T K_{T-1} A E\{x_{T-2}|I_{T-2}\} \end{aligned}$$

We can proceed similarly to obtain the optimal policy for every stage:

$$u_t(I_t) = \mu^*_t(I_t) = L_t E\{x_t|I_t\}$$

where the matrix L_t is given by

$$L_t = -(1-\alpha)(R_t + (1-\alpha)B^T K_{t+1} B)^{-1} B^T K_{t+1} A$$

with the matrices K_t given recursively by the Riccati equation, starting with $K_T = Q_T$:

$$\begin{aligned} P_t &= (1-\alpha)^2 A^T K_{t+1} B (R_t \\ &+ (1-\alpha) B^T K_{t+1} B)^{-1} B^T K_{t+1} A \\ K_t &= A^T K_{t+1} A - P_t + Q_t \end{aligned}$$

Thus, the total cost has two separate components, one due to control and the other due to estimation. Note that the optimal controller is a function of only the packet loss probability of the uplink, α. Link failure probability of the downlink, β, has no affect on the control, but it does affect the optimal cost through estimation.

The cost associated with the state estimation error, from t through T is:

$$\sum_{\tau=t}^{T-1} E\{(x_\tau - E[x_\tau|I_\tau])^T P_\tau (x_\tau - E[x_\tau|I_\tau])\}. \quad (15)$$

Let $e_t = x_t - E[x_t|I_t]$. Information theoretically, the minimum rate required for the encoder to send information to a receiver with side information is achieved by transmitting the conditioned information [13]. Furthermore, since the system is Markov, the sufficient statistic about the density for encoding is captured by the latest outcome, and the current information at the receiver's memory [8]. Suppose the encoder at each stage encodes the innovation (state conditioned on the information available at the controller), ξ, over the channel:

$$\xi_{t+1} = x_{t+1} - AE[x_t|I_t] - \alpha_t B u_t = A(e_t).$$

Suppose the quantizer that we seek has a fixed number, K, of cells (bins). Let $e^j, j = 1,\ldots,n$ be the components of the state estimation error vector, B_i be the ith quantization bin; $B_i = \{x : x \in B_i\}, i = 1,\ldots,K$ and $f_i = f(x|x \in B_i), i = 1,\ldots,K$ be the conditional probability density for the ith bin. Then, the evolution in the estimation error would be

$$T(f(e_t^1,\ldots,e_t^n)|e_t \in B_i, \beta_t = 1) = (\frac{1}{|A|} f_i(A^{-1}(x - q_i)),$$

$$T(f(e_t^1,\ldots,e_t^n)|e_t \in B_i, \beta_t = 0) = (\frac{1}{|A|} f_i(A^{-1}(x)),$$

where the first line takes place when there is a successful transmission from the plant to the controller and the second happens when there is a failure in the channel.

Let V_{t-1} denote the state estimation error cost from stage $t-1$ through T. The optimal quantizer at time $t-1$ would satisfy the following DP equation.

$$\begin{aligned} V_{f(e_{t-1})} &= \min_{Q_{t-1}} E[e_{t-1}^T P_{t-1} e_{t-1} \\ + \sum_{i=1}^{K} p(e_{t-1} \in B_i) \Big(&(1-\beta) V_{T(f(e_t)|e_{t-1} \in B_i, \beta_{t-1}=1)} \\ &+ \beta V_{T(f(e_t)|e_{t-1} \in B_{i,t}, \beta_{t-1}=0)} \Big)], \end{aligned} \quad (16)$$

However, this solution does not provide much insight. As a suboptimal but a practical solution, one could use *sequential quantization of scalar components*. In this case, since asymptotically the errors are known to converge to uniform densities, and a uniform quantizer applied to a uniform density regenerates uniform error, the quantizer minimizing the one-stage myopic cost is identical to the quantizer (up to a scaling update) minimizing the long-run cost. Thus, an analysis similar to Proposition 2.5 should be followed; let K_i be given by (6), then the estimation error for each component will be independent and of the form

$$E[e_t^{i\,2}] = \frac{1}{1 - (|\lambda_i|^2(\beta + (1-\beta)/K_i^2)} \Delta_0^{i\,2}.$$

B. *Mean-Square Stability of State*

In this subsection we find the conditions on the controller and the uniform quantizer which would achieve mean-square stability in the state. We will consider scalar systems, which can be interpreted as the decoupled components of a vector. We will restrict the controller to be linear and the quantizer to be uniform, following the results of the previous sections. We also include a case where the latest available control is used, in addition to the case where zero control is used; in both cases the control is assumed to be linear in its state estimate.

1) *Case where zero control is used in case of a packet loss:* Consider the case where zero control is used in case of a packet loss, which occurs with probability β.

Defining a two-dimensional state $Y_t = \begin{bmatrix} E[(x)_t^2] \\ E[(e)_t^2] \end{bmatrix}$, with probability $1-\beta$ the mean-square evolution will be $Y_{t+1} = BY_t$, where b is

$$B := \begin{bmatrix} (a+bk)^2 & -bk(bk+2a) \\ 0 & a^2/K^2 \end{bmatrix}.$$

In case of a packet loss, the system evolution will be $Y_{t+1} = AY_t$, where A is

$$A = \begin{bmatrix} a^2 & 0 \\ 0 & a^2 \end{bmatrix}.$$

These together lead to a Bernoulli random evolution of the expanded state, Y. Defining $M := (\beta A + (1-\beta)B)$, we have

$$E[Y_{t+1}] = ME[Y_t].$$

Thus, the system will be stable if;

$$a\beta^2 + (1-\beta)\max((a+bk)^2, a^2/K^2) < 1. \qquad (17)$$

Note that, to achieve stability $a\beta^2$ should always be less than 1, as a necessary condition.

2) Case where latest available control is used: Defining a four-dimensional state

$$Y_t = \begin{bmatrix} E[x_t'^2] \\ E[u_{t-1}^2] \\ E[x_t u_{t-1}] \\ E[e_t^2] \end{bmatrix},$$

with probability $1-\beta$ the system evolution will be $Y_{t+1} = BY_t$, where B is

$$B = \begin{bmatrix} (a+bk)^2 & 0 & 0 & -bk(2a+b+k) \\ k^2 & 0 & 0 & -k^2 \\ (a+b+k)k & 0 & 0 & -(a+b+k)k \\ 0 & 0 & 0 & a^2/K^2 \end{bmatrix}.$$

On the other hand, in case of a packet loss, the system equation will be $Y_{t+1} = AY_t$, where A is

$$A = \begin{bmatrix} a^2 & b^2 & 2ab & 0 \\ 0 & 1 & 0 & 0 \\ 0 & b & a & 0 \\ 0 & 0 & 0 & a^2 \end{bmatrix}.$$

Again, there will be a Bernoulli random evolution of the expanded state, Y. Let $N := \beta A_i + (1-\beta)b$. Then, we have

$$E[Y_{t+1}] = NE[Y_t], \qquad (18)$$

which is stable if all the eigenvalues of N, λ_j, are within the unit circle.

IV. Conclusions

This paper has investigated the problem of remotely controlling an LTI plant over communication links. In particular, we have established the rate requirements to achieve stability under different criteria. Also, we showed that the optimal control problem with a quadratic cost has a solution that is separable into a control and an estimation part, where the estimation part alone affects the communication rate requirements when the measured state is quantized. Optimal controller and encoders are provided.

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