Structural Solutions via Optimal Reverse-Waterfilling Algorithms in Low-Delay Quantized MMSE State Estimation

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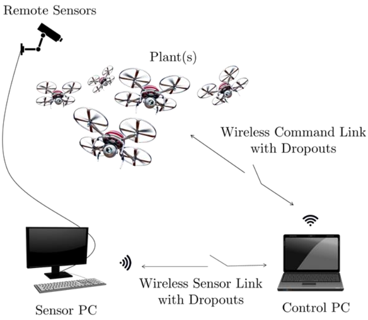
Fig. 1: Network Control System (NCS)

Fundamental Questions: (i) How do the network-induced delay, packet loss, quantization errors, and communication channel affect the stability of the system; (ii) Under what conditions is an NCS stabilizable, and how does one stabilize it? (iii) What are the performance limitations in an NCS and how does its synthesis affects the corresponding performance criteria?



[Tatikonda-Sahai-Mitter:2004]

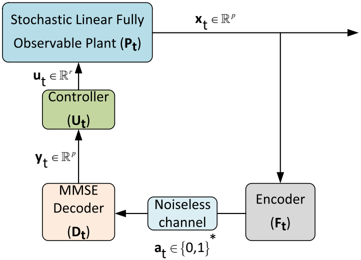


Fig. 2: NCS with a single-loop

Goals:

- Derive a control theoretic separation principle for the quantized LQG control with á priori structure of the quantizer
- Minimize the infinite horizon LQG cost, i.e., $\limsup_{t\longrightarrow\infty} \frac{1}{n+1}\sum_{t=0}^{n} \mathbf{E}\{||\mathbf{x}_t||_Q^2 + ||\mathbf{u}_t||_R^2\}$, $Q \succ 0, R \succ 0$

Stochastic Linear Plant

$(\mathbf{P}_t)$ $\mathbf{x}_{t+1} = A\mathbf{x}_t + B\mathbf{u}_t + \mathbf{w}_t$, $A \in \mathbb{R}^{p\times p}$, $B \in \mathbb{R}^{p\times r}$, $(A,\ B)$ controllable pair, $\mathbf{w}_t \sim \mathcal{N}(0;\Sigma_{\mathbf{w}})$, $\Sigma_{\mathbf{w}} \succ 0$, i.i.d., indep. of $\mathbf{x}_0$

Noiseless Channel*

$$\mathbf{a}_t \subset \{0,1,00,01,\ldots\},\ \forall t \qquad (1)$$

Encoder/Decoder

$(\mathbf{E}_t)$ $\mathbf{P}(da_t|a^{t-1},x^t,y^{t-1},u^{t-1})$
$(\mathbf{D}_t)$ $\mathbf{P}(dy_t|a^t,y^{t-1},u^{t-1})$

Controller

$(\mathbf{U}_t)$ $\mathbf{P}(du_t|y_t)$

* An additive Vector Gaussian noise channel is also assumed in place of the noiseless channel

[Tatikonda-Sahai-Mitter:2004] (cont'd)

Control theoretic separation principle

Á priori chosen the encoder to be an innovations or predictive quantizer. Then,

$$\mathbf{u}_t(y_t) = L\mathbf{y}_t, \mathbf{y}_t = A\mathbf{y}_{t-1} + B\mathbf{u}_{t-1}(y_{t-1}) + \mathbf{I}_t, \tag{2}$$

where $\mathbf{y}_t = \mathbf{E}\{\mathbf{x}_t|\mathbf{a}^t\}$ is the decoder's output; $\mathbf{I}_t$ is the innovation process (independent of the control signals); $L \in \mathbb{R}^{r\times p}$ is the control (feedback) gain
The (linear) controller achieves the following cost

$$LQG_\infty^* = \underbrace{\text{trace}(\Sigma_{\mathbf{w}}K)}_{\text{control cost}} + \underbrace{\text{trace}(\Theta\Delta)}_{\text{quantized state estimation}}, \tag{3}$$

where $\Theta = A^{\mathrm{T}}KA - K + Q \succ 0$, $K \succeq 0$ is the unique stabilizing solution of the DARE

$$K = A^{\mathrm{T}}KA - A^{\mathrm{T}}KB(B^{\mathrm{T}}KB + R)^{-1}B^{\mathrm{T}}KA + Q, \tag{4}$$

Δ is the time-invariant counterpart of $\Delta_t = \mathbf{E}\left\{(\mathbf{x}_t - \mathbf{y}_t)(\mathbf{x}_t - \mathbf{y}_t)^{\mathrm{T}}\right\}$

(i) A tight lower bound on (3) can be achieved via nonanticipative rate distortion theory [Gorbunov-Pinsker:1972]
(ii) A closed form expression of a lower bound on (3) for scalar processes
(iii) A lower bound on the quantized state estimation problem for multivariate processes obtained via a sub-optimal reverse-waterfilling solution

- Mean-square stability conditions for linear dynamical systems are provided for instance in [Baillieul:1999,2001], [Wong-Brockett:1999], [Brockett-Liberzon: 2000], [Hespanha-Ortega-Vasudevan: 2002], [Tatikonda-Mitter:2004], [Nair-Evans:2004]
- Further studies regarding the separation principle in quantized LQG control are given in [Bao-Skoglund-Johansson:2011], [Fu:2012], [Yuksel:2014]
- Average data rate bounds of single loop SIMO NCS using directed information, random or fixed delays in the system and ECDQ coding schemes are studied in [Silva-Derpich-Østergaard:2011], [Silva-Derpich-Østergaard-Encina:2016], [Barforooshan-Derpich-Stavrou-Østergaard:2020]
- The setup of Tatikonda was revisited in [Tanaka-Kim-Parrilo-Mitter:2017], [Tanaka-Esfahani-Mitter:2018]. Main results include: (i) the exact computation of the nonanticipative RDF via an SDP algorithm for multivariate Gauss-Markov processes, (ii) the solution and a realization of the rate-cost function for Gaussian processes via a three steps design comprised of a feedback controller design, a virtual sensor, Kalman filter
- *Achievability schemes* for the quantized state estimation or the closed-loop control setup of Tatikonda are explored in [Tanaka-Johansson-Oechtering-Sandberg-Skoglund:2016], [Stavrou-Østergaard-Charalambous:2018], [Kostina-Hassibi:2019]
- General *lower and upper bounds for fully observable system models beyond additive* i.i.d. *Gaussian processes and partially observable systems driven by Gaussian noise* are studied in [Kostina-Hassibi:2019]

Case 1: Fully Observable Gauss-Markov Process with MSE Distortion

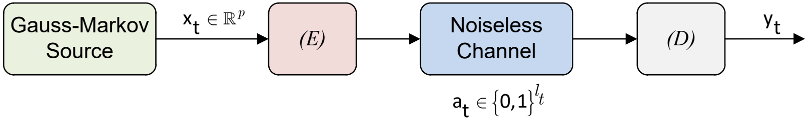


Fig. 3: Low-delay quantization system

- **Uncontrolled Gauss-Markov Source**:

$$\mathbf{x}_{t+1} = A\mathbf{x}_t + \mathbf{w}_t,\ t \in \mathbb{N}_0,\ \mathbf{x}_0 \in \mathbb{R}^p \sim \mathcal{N}(0; \Sigma_{\mathbf{x}_0}) \tag{5}$$

  1. $A \in \mathbb{R}^{p\times p}$ is a non-random (known) matrix
  2. $\mathbf{w}_t \in \mathbb{R}^p \sim \mathcal{N}(0; \Sigma_{\mathbf{w}})$ i.i.d. sequence, $\Sigma_{\mathbf{w}} \succ 0$, independent of $\mathbf{x}_0$.

- **Low-delay processing of information**:

$$(\mathcal{E}):\ a_t = f_t(a^{t-1}, x^t),\quad (\mathcal{D}):\ y_t = g_t(a^t); \tag{6}$$

  1. initial time ($t = 0$): $z_0 = f_0(x_0)$ and $y_0 = g_0(z_0)$
  2. clocks of the encoder/decoder are synchronized

- **Empirical rates**: For $D > 0$, we define:

$$R^{\text{op}}(D) \triangleq \inf_{\substack{(f_t,\ g_t):\ t=0,1,\ldots,\infty \\ \limsup_{n\longrightarrow\infty} \frac{1}{n+1}\sum_{t=0}^{n} \mathbb{E}\{||\mathbf{x}_t - \mathbf{y}_t||_2^2\} \leq D}} \limsup_{n\longrightarrow\infty} \frac{1}{n+1}\sum_{t=0}^{n} H(\mathbf{y}_t|\mathbf{y}^{t-1}) \tag{7}$$

Lower bound on $R^{\text{op}}(D)$

$$R^{\text{op}}(D) \geq R^{\text{na}}(D) = \inf_{\substack{\mathbf{P}(dy_t|y^{t-1},x_t):\ t=0,1\dots,\infty \\ \limsup_{n\longrightarrow\infty} \frac{1}{n+1}\sum_{t=0}^{n}\mathbf{E}\{||\mathbf{x}_t-\mathbf{y}_t||_2^2\}\leq D}} \limsup_{n\longrightarrow\infty} I(\mathbf{x}^n \to \mathbf{y}^n) \quad (8)$$

where $I(\mathbf{x}^n \to \mathbf{y}^n) \triangleq \frac{1}{n+1}\sum_{t=0}^{n} I(\mathbf{x}_t;\mathbf{y}_t|\mathbf{y}^{t-1})$

Time-Invariant Characterization for jointly Gaussian process*

- $\{(\mathbf{x}_t,\mathbf{y}_t):\ t=0,1,\dots\}$ is jointly Gaussian
- $\mathbf{P}(dy_t|y^{t-1},x_t) \equiv \mathbf{P}^G(dy_t|y_{t-1},x_t)$ can be realized as follows

$$\mathbf{y}_t = H\mathbf{x}_t + (I_p - H)A\mathbf{y}_{t-1} + \mathbf{v}_t,\ I_p \text{ is the identity matrix; } \mathbf{v}_t \sim \mathcal{N}(0;\Sigma_{\mathbf{v}}) \quad (9)$$

- $(H,\Sigma_{\mathbf{v}})$ are designing matrices chosen such **(i)** $\widehat{\mathbf{x}}_{t|t} \equiv \mathbf{E}\{\mathbf{x}_t|\mathbf{y}^t\} = \mathbf{y}_t - a.s$, **(ii)** $\limsup_{n\longrightarrow\infty}\frac{1}{n+1}\sum_{t=0}^{n}\mathbf{E}\{||\mathbf{x}_t-\mathbf{y}_t||_2^2\} = \text{trace}(\Delta)$ with

$$H = I_p - \Delta\Lambda^{-1},\quad \Sigma_{\mathbf{v}} = \Delta H^{\mathrm{T}} \quad (10)$$

  $(\Delta,\ \Lambda)$ are the time-invariant values of $(\Sigma_{t|t},\Sigma_{t|t-1})$.
- For $D>0$, the optimization problem to solve parametrized by $(H,\Sigma_{\mathbf{v}})$ is

$$R^{\text{na}}(D) = \min_{\substack{0\prec\Delta\preceq\Lambda \\ \text{trace}(\Delta)\leq D}} \frac{1}{2}\log\left(\frac{\det(\Lambda)}{\det(\Delta)}\right) \quad (11)$$

* This is derived in [Stavrou-Charalambous-Charalambous-Loyka-Skoglund:2018]

Proposition 1: Structural result

Let $(A, \Sigma_{\mathbf{w}})$ admit one of the following strong structural properties:

1. $A \in \mathbb{R}^{p \times p}$ is real symmetric matrix and $\Sigma_{\mathbf{w}} = \sigma_{\mathbf{w}}^2 I_p$ (scalar symmetric)
2. $A = \alpha I_p$ (scalar symmetric) and $\Sigma_{\mathbf{w}} \succ 0$
3. Both $A = \Sigma_{\mathbf{w}} \succ 0$
4. Both $(A, \Sigma_{\mathbf{w}})$ are diagonal matrices (trivial).

Then, $(A, \Sigma_{\mathbf{w}}, \Delta)$ commute by pairs and consequently (Λ, Δ) commute in (11)

Proof. Makes use of the "eigenvector alignment" between the design variable Δ and either A or $\Sigma_{\mathbf{w}}$, depending on which one is real symmetric (the other is scalar matrix).

Simplified expression of (11)

Suppose that one of the previous structural properties of (A, Σ_W) hold. Then (11) achieves smaller rates and simplifies to

$$R^{\text{na}}(D) = \inf_{\substack{0 < \mu_{\Delta,i} \leq \mu_{\Lambda,i},\ i=1,\ldots,p \\ \sum_{i=1}^{p} \mu_{\Delta,i} \leq D}} \frac{1}{2} \sum_{i=1}^{p} \log\left(\frac{\mu_{\Lambda,i}}{\mu_{\Delta,i}}\right), \tag{12}$$

with $\mu_{\Lambda,i} = \mu_{A^2,i}\mu_{\Delta,i} + \mu_{\Sigma_W,i}$.

Proof. Makes use of Hadamard's inequality that under the strong structural properties of $(A, \Sigma_{\mathbf{w}})$ holds with equality.

Optimal reverse-waterfilling solution

The optimal parametric solution of (12) is the following

$$R^{\mathrm{na}}(D) = \frac{1}{2}\sum_{i=1}^{p} \log\left(\frac{\mu_{\Lambda,i}}{\mu_{\Delta,i}}\right), \tag{13}$$

where $\mu_{\Delta,i}$ is computed by the reverse-waterfilling algorithm

$$\mu_{\Delta,i} = \begin{cases} \xi_i, & \text{if } \xi_i < \mu_{\Lambda,i}, \\ \mu_{\Lambda,i}, & \text{otherwise} \end{cases}, \ \forall i, \tag{14}$$

and $\xi_i > 0$ is computed as follows

$$\xi_i = \frac{1}{2\mu_{B,i}}\left(\sqrt{1+\frac{2\mu_{B,i}}{\theta}} - 1\right), \ \mu_{B,i} \neq 0, \tag{15}$$

$$\xi_i = \frac{1}{2\theta}, \ \mu_{B,i} = 0, \tag{16}$$

with $\mu_{B,i} \triangleq \frac{\mu_{A^2,i}}{\mu_{\Sigma_{\mathbf{w}},i}}$ and $\theta > 0$ chosen such that $\sum_{i=1}^{p} \mu_{\Delta,i} = D$.

Proof. Invoke and solve KKT conditions (necessary and sufficient conditions for global optimality).

Algorithm 1 Implementation of the reverse-waterfilling solution

Initialize: number of p; D; ϵ; nominal minimum and maximum θ, i.e, $\theta^{\min}$ and $\theta^{\max}$; initial variance for $\mu_{\Lambda,1}$; pick the matrix structure of $(A,\ \Sigma_{\mathbf{w}})$ in (5) and compute their corresponding eigenvalues $\{(\mu_{A,i}, \mu_{\Sigma_{\mathbf{w}},i}) :\ i \in \mathbb{N}_1^p\}$ (in increasing or decreasing order).
Set $\theta = p/2D$; flag $= 0$.
while flag $= 0$ **do**
  Compute $\mu_{\Delta,i}$ $\forall$ i as follows:
  for $i = 1 : p$ **do**
    Compute ξ_i according to (15) or (16).
    Compute $\mu_{\Delta,i}$ according to (14).
  end for
  if $\sum_{i=1}^{p} \mu_{\Delta,i} - D \geq \epsilon$ **then**
    Set $\theta^{\min} = \theta$
  else
    Set $\theta^{\max} = \theta$
  end if
  if $\theta^{\max} - \theta^{\min} \geq \epsilon$ **then**
    Compute $\theta = \frac{(\theta^{\min} + \theta^{\max})}{2}$
  else
    flag $\leftarrow 1$
  end if
end while
Output: $\{\mu_{\Delta,i} :\ i = 1, \ldots, p\}$, $\{\mu_{\Lambda,i} :\ i = 1, \ldots, p\}$, for a given distortion level D.

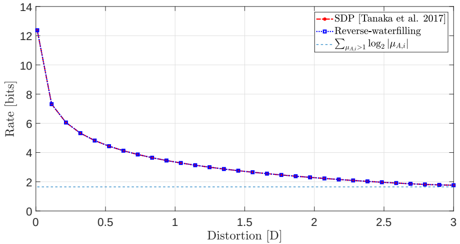
[Comparison with [Tanaka-Kim-Parrilo-Mitter:2017]

Input data

Consider a time-invariant Gauss-Markov process with $(A, \Sigma_{\mathbf{w}})$ given by

$$A = \begin{bmatrix} 1.1016 & 1.2190 & 0.4165 \\ 1.2190 & 1.7859 & 1.1035 \\ 0.4165 & 1.1035 & 0.1029 \end{bmatrix}, \quad \Sigma_{\mathbf{w}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{17}$$

and $D \in [0.01, 3]$.



- Minimum data rates of mean-square stability are ≈ 1.6493 bits/vector source

Comparison with [Tanaka-Kim-Parrilo-Mitter:2017] (cont'd)

Input Data: Choose a pair $(A, \Sigma_{\mathbf{w}})$ that satisfy the strong structural properties of Proposition 1; $D \in (0, \infty)$.

Solver (Numb. dimens. $p = 10$)	**Mean**
SDP Algorithm (by default $\epsilon = 10^{-9}$)	0.701
Algorithm 1 ($\epsilon = 10^{-9}$)	2.01×10^{-4}
Solver (Numb. dimens. $p = 50$)	**Mean**
SDP Algorithm (by default $\epsilon = 10^{-9}$)	122.017
Algorithm 1 ($\epsilon = 10^{-9}$)	2.61×10^{-4}
Solver (Numb. dimens. $p = 1000$)	**Mean**
SDP Algorithm (by default $\epsilon = 10^{-9}$)	non-conclusive
Algorithm 1 ($\epsilon = 10^{-9}$)	3.26×10^{-4}

Table I: Average omputational time for execution between SDP algorithm and Algorithm 1 for 1000 instances.

1. For $p = 10$, Algorithm 1 executes 3000 times faster than SDP algorithm;
2. For $p = 50$, Algorithm 1 executes $\approx$ 450000 times faster than SDP algorithm;
3. For $p = 1000$ Algorithm 1 is very very fast while SDP result is inconclusive because it executes very very slow (it takes days to operate)

- Algorithm 1 is much much faster compare to SDP algorithm and more importantly scalable $\Longrightarrow$ Desirable in computationally limited systems

Closed form solution beyond scalar processes

Input data

Consider a time-invariant Gauss-Markov process with $(A, \Sigma_{\mathbf{w}})$ given by

$$A = \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}, \quad \Sigma_{\mathbf{w}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{18}$$

and $D > 0$. Let $\mu_{A,1} = 1.5 \geq \mu_{A,2} = 0.5$ (decreasing order).

Analytical expression

$$R^{\text{na}}(D) = \frac{1}{2}\left[\log\left(\frac{9}{4} + \frac{8}{\sqrt{(D+4)(9D+4)} - (D+4)}\right) + \log\left(\frac{1}{4} + \frac{8}{(9D+4) - \sqrt{(D+4)(9D+4)}}\right)\right], \tag{19}$$

$$R^{\text{na}}(D) = \frac{1}{2}\log\left(\frac{9}{4} + \frac{1}{\frac{2}{9}\left(\sqrt{1 + \frac{(9D-8)(9D-12)}{4}} - 1\right)}\right), \tag{20}$$

1. (19) corresponds to the full rank solution (both dimensions are active).
2. (20) corresponds to the rank deficient solution (only the first dimension is active). In fact, for $D \longrightarrow \infty$, (20) gives $R^{\text{na}}(D) \approx 0.585$ bits/source sample (minimum data rate for MS stability)

Upper bound on $R^{\mathrm{op}}(D)$

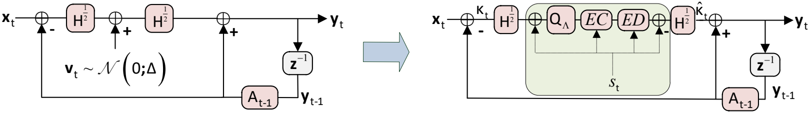


Fig. 4: Optimal realization that achieves the lower bound $R^{\mathrm{na}}(D)$ (left figure) and the same realization with the additive Gaussian noise replaced by coding noise via ECDQ scheme (right figure)

1. ECDQ scheme "simulates" the optimal minimizer of the quadratic Gaussian RDF, i.e., $R^{\mathrm{na}}(D)$
2. Lattice code can be seen as the counterpart of linear codes in Euclidean space (these are structural codes)
3. "Dithering" s_t is a randomization (noise!) added to guarantee desired distortion independent of the input statistics and improve the quantization, it is of particular help at low rates

Description of the ECDQ scheme

- *Encoder*: receives the innovations of $\mathbf{x}_t$, i.e., $\boldsymbol{\kappa}_t = \mathbf{x}_t - A\mathbf{y}_{t-1},\ \mathbf{y}_{t-1} = \mathbf{E}\{\mathbf{x}_{t-1}|\mathbf{a}^{t-1}\}$ and quantizes $H^{\frac{1}{2}}\boldsymbol{\kappa}_t + s_t$ using a $p-$dimensional lattice, i.e., $\mathbf{a}_t = Q_\Lambda(H^{\frac{1}{2}}\boldsymbol{\kappa}_t + \mathbf{s}_t),\ H^{\frac{1}{2}} = \sqrt{I_p - \Delta\Lambda^{-1}}$
- *Decoder*: receives the coded bits and generates as estimate by subtracting $\mathbf{s}_t$ from quantizer's output multiplied by the scaling $H^{\frac{1}{2}}$, $\hat{\boldsymbol{\kappa}}_t = H^{\frac{1}{2}}(Q_\Lambda(H^{\frac{1}{2}}\boldsymbol{\kappa}_t + \mathbf{s}_t) - \mathbf{s}_t)$

Upper bound on $R^{\text{op}}(D)$ (cont'd)

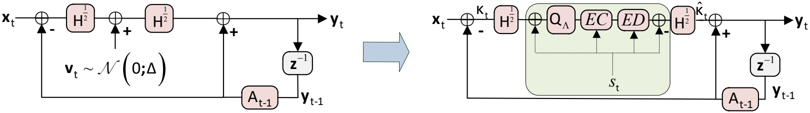


Fig. 5: Optimal realization that achieves the lower bound $R^{\text{na}}(D)$ (left figure) under the proposed strong structural properties and the same realization with the additive Gaussian noise replaced by coding noise via ECDQ scheme (right figure)

Performance Analysis

- *MSE Distortion*:
  $D = \mathbf{E}\{||\mathbf{x}_t - \mathbf{y}_t||_2^2\} = \mathbf{E}\{||\boldsymbol{\kappa}_t - \hat{\boldsymbol{\kappa}}_t||\} = \mathbf{E}\{||H^{\frac{1}{2}}(Q_\Lambda(H^{\frac{1}{2}}\boldsymbol{\kappa}_t + \mathbf{s}_t) - \mathbf{s}_t) - \boldsymbol{\kappa}_t||_2^2\}, \quad (\star)$
- *Coding Rate*:

$$\lim_{n\longrightarrow\infty} \frac{1}{n+1}\sum_{t=0}^{n} H(Q_\Lambda|\mathbf{s}_t) \leq R^{\text{na}}(D) + \underbrace{\frac{r}{2}\log\left(2\pi e G_r\right)}_{\text{rate loss}}, \quad r \triangleq \text{rank}(H) \tag{21}$$

1. For $r=1$, $G_1 = 1/12$ and the rate loss is approx. 0.254 bits/source sample;
2. If r increases, the rate loss becomes smaller and smaller; if $r \longrightarrow \infty$ $G_\infty = \frac{1}{2\pi e}$ and the coding noise is exactly the Gaussian noise.

*This approach can be found in the literature as DPCM based ECDQ scheme, e.g., [Stavrou-Østergaard-Charalambous:2018],[Khina-Kostina-Khisti-Hassibi:2019]

Open Question: Bounds for LQG Control

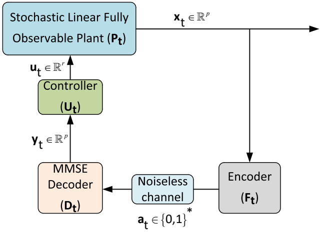


Fig. 6: NCS with a single-loop

Operational Rate Cost Function*

$$R^{\text{op}}(\Gamma) = \inf_{\substack{(\mathbf{E}_t,\mathbf{D}_t,\mathbf{U}_t):\ t=0,\ldots,\infty \\ \limsup_{n\longrightarrow\infty}\frac{1}{n+1}LQR(\mathbf{x}^n,\mathbf{u}^{n-1})\leq\Gamma}} \limsup_{n\longrightarrow\infty}\frac{1}{n+1}\sum_{t=0}^{n} H(\mathbf{y}_t|\mathbf{y}^{t-1},\mathbf{u}^{t-1})^* \quad (22)$$

where $LQR(\mathbf{x}^n,\mathbf{u}^{n-1}) \triangleq \mathbf{E}\left[\sum_{t=0}^{n-1}\{||\mathbf{x}_t||_Q^2 + ||\mathbf{u}_t||_R^2\} + ||\mathbf{x}_n||_Q^2\right]$

1. For deterministic controllers, i.e., $u_t = e_t(y^t)$, it can be shown that $H(\mathbf{y}_t|\mathbf{y}^{t-1},\mathbf{u}^{t-1}) \equiv H^e(\mathbf{y}_t|\mathbf{y}^{t-1})$ meaning that the distribution of the conditional entropy is specified once the control signals are specified
2. For co-located decoder/controller we can take $H(\mathbf{u}_t|\mathbf{u}^{t-1})$

Open Question: Bounds for LQG control (cont'd)

Lower Bound: General Rate Cost Function*

$$R^{\mathrm{na}}(\Gamma) = \inf_{\substack{\mathbf{P}(dy_t|y^{t-1},u^{t-1},x^t):\ t=0,\ldots,\infty \\ \limsup_{n\longrightarrow\infty} \frac{1}{n+1} LQR(\mathbf{x}^n,\mathbf{u}^{n-1})\leq\Gamma}} \limsup_{n\longrightarrow\infty} \frac{1}{n+1} I(\mathbf{x}^n \rightarrow \mathbf{y}^n || \mathbf{u}^{n-1})^* \quad (23)$$

$I(\mathbf{x}^n \rightarrow \mathbf{y}^n||\mathbf{u}^{n-1}) \triangleq \sum_{t=0}^{n} I(\mathbf{x}^t;\mathbf{y}_t|\mathbf{y}^{t-1},\mathbf{u}^{t-1})$

1. For deterministic controller, i.e., $u_t = e_t(y^t)$, it can be shown that $I(\mathbf{x}^n \rightarrow \mathbf{y}^n||\mathbf{u}^{n-1}) \equiv I(\mathbf{x}^n \rightarrow \mathbf{y}^n)$
2. For co-located decoder/controller we can take $I(\mathbf{x}^n \rightarrow \mathbf{u}^n)$ [Tanaka-Esfahani-Mitter:2018]
3. Characterization of the (Gaussian) rate-cost function using the separation principle see, e.g., [Tatikonda-Sahai-Mitter:2004] or [Tanaka-Esfahani-Mitter:2018]

$$\boxed{R(\Gamma) = \min_{\substack{0\prec\Delta\preceq\Lambda \\ \mathrm{trace}(\Sigma_{\mathbf{w}}K)+\mathrm{trace}(\Theta\Delta)\leq\Gamma}} \frac{1}{2}\log\left(\frac{\det(\Lambda)}{\det(\Delta)}\right)} \quad (24)$$

where $\Theta = A^{\mathrm{T}}KA - K + Q \succ 0$

Open Question: Bounds for LQG control (cont'd)

Now (24) is equivalent to the lower bound of the quantized state estimation problem

$$R^{\text{na}}(\bar{D}) = \min_{\substack{0 \prec \Delta \preceq \Lambda \\ \text{trace}(\Theta\Delta) \leq \bar{D}}} \frac{1}{2} \log \left(\frac{\det(\Lambda)}{\det(\Delta)} \right) \tag{25}$$

where $\bar{D} \triangleq \Gamma - \text{trace}(\Sigma_{\mathbf{w}} K) > 0$. (25) is achieved by a linear realization of the form [Tatikonda-Sahai-Mitter:2004], [Stavrou-Charalambous-Charalambous-Loyka:2018]

$$\begin{aligned} \mathbf{y}_t &= A\mathbf{y}_{t-1} + \mathbf{u}_{t-1}(y_{t-1}) + I_t, \\ &= A\mathbf{y}_{t-1} + \mathbf{u}_{t-1}(y_{t-1}) + H(\mathbf{x}_t - A\mathbf{y}_{t-1} - \mathbf{u}_{t-1}(y_{t-1})) + \mathbf{v}_t \end{aligned} \tag{26}$$

where $(H, \Sigma_{\mathbf{v}})$ are obtained in closed form similar to the realization of the quantized state estimation problem, i.e.,

$$H = I_p - \Delta\Lambda^{-1}, \quad \Sigma_{\mathbf{v}} = \Delta H^{\text{T}} \tag{27}$$

The complete realization that corresponds to (24) can be obtained using the fact that $\mathbf{u}_t = L\mathbf{y}_t$ [Tatikonda-Sahai-Mitter:2004]

1. Can we find an optimal reverse-waterfilling solution to the characterization of (25)?
2. Can we find optimal closed form expressions beyond scalar processes?
3. Extension to time-varying processes?

Open Question: Bounds for LQG control (cont'd)

Upper Bound on $R^{\text{op}}(\Gamma)$

$$R^{\text{op}}(\Gamma) \leq R^{\text{na}}(\bar{D}) + \frac{r}{2}\log(2\pi e G_r) \tag{28}$$

where $r = \text{rank}(H)$

1. (28) is achieved using precisely the ECDQ scheme of the quantized state estimation problem
2. Fundamental difference with quantized state estimation problem is that at the innovations encoder we subtract the previous control signals and add them at the decoder following precisely the realization of the lower bound

Case 2: Partially Observable Gauss-Markov Process with MSE Distortion

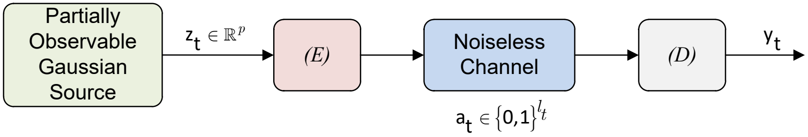


Fig. 7: Low-delay quantization system

- **Uncontrolled Partially Observable Gauss-Markov Source**:

$$\begin{aligned} \mathbf{x}_{t+1} &= A\mathbf{x}_t + \mathbf{w}_t,\ t \in \mathbb{N}_0,\ \mathbf{x}_0 \in \mathbb{R}^p \sim \mathcal{N}(0;\Sigma_{\mathbf{x}_0}) \\ \mathbf{z}_t &= C\mathbf{x}_t + \mathbf{n}_t \end{aligned} \tag{29}$$

  1. $A \in \mathbb{R}^{p\times p}, C \in \mathbb{R}^{m\times p}, m \leq p$, full row rank both non-random (known) matrices
  2. $\mathbf{w}_t \in \mathbb{R}^p \sim \mathcal{N}(0;\Sigma_{\mathbf{w}}), \mathbf{n}_t \in \mathbb{R}^m \sim \mathcal{N}(0;\Sigma_{\mathbf{n}})$ are both i.i.d. sequences, independent of each other with $\Sigma_{\mathbf{w}} \succ 0, \Sigma_{\mathbf{n}} \succeq 0$, independent of $\mathbf{x}_0$.
- **Low-delay processing of information**:

$$(\mathcal{E}):\ a_t = f_t(a^{t-1}, z^t), \quad (\mathcal{D}):\ y_t = g_t(a^t); \tag{30}$$

  1. initial time ($t=0$): $z_0 = f_0(x_0)$ and $y_0 = g_0(z_0)$
  2. clocks of the encoder/decoder are synchronized
- **Empirical rates**: For $D>0$, we define:

$$R_{\text{in}}^{\text{op}}(D) \triangleq \inf_{\substack{(f_t,\ g_t):\ t=0,1,\ldots,\infty \\ \limsup_{n\longrightarrow\infty} \frac{1}{n+1}\sum_{t=0}^{n}\mathbb{E}\{||\mathbf{x}_t-\mathbf{y}_t||_2^2\}\leq D}} \limsup_{n\longrightarrow\infty} \frac{1}{n+1}\sum_{t=0}^{n} H(\mathbf{y}_t|\mathbf{y}^{t-1}) \tag{31}$$

Lower bound on $R_{\text{in}}^{\text{op}}(D)$

Data Processing inequality

$$R_{\text{in}}^{\text{op}}(D) \geq R_{\text{in}}^{\text{na}}(D) = \inf_{\substack{\mathbf{P}(dy_t|y^{t-1}, z^t):\ t=0,1\ldots,\infty \\ \limsup_{n\longrightarrow\infty} \frac{1}{n+1}\sum_{t=0}^{n}\mathbf{E}\{||\mathbf{x}_t - \mathbf{y}_t||_2^2\} \leq D}} \limsup_{n\longrightarrow\infty} I(\mathbf{z}^n \rightarrow \mathbf{y}^n) \tag{32}$$

where $I(\mathbf{z}^n \rightarrow \mathbf{y}^n) \triangleq \frac{1}{n+1}\sum_{t=0}^{n} I(\mathbf{z}^t; \mathbf{y}_t|\mathbf{y}^{t-1})$

- Solving precisely (32) is still an open problem
- Remarkable efforts to solve the lower bound have been made in [Tanaka:2015], [Tanaka-Esfahani-Mitter:2018], [Kostina-Hassibi:2019]
  1. [Tanaka:2015] considers a different optimization problem than (32) (with soft distortion constraints);
  2. [Tanaka-Esfahani-Mitter:2018] considered a complicated structural result where a variant of (32) is studied, i.e., the partially observable process is reduced to a fully observable via a pre-Kalman filtering approach (sending an estimate of the indirectly observed process) with a modified cost
  3. [Kostina-Hassibi:2019] address the problem by reducing it to a modified fully observable Gauss-Markov process driven by the covariance of its innovations process (it requires the some computations of a pre-Kalman filter); They arrived to some closed form expressions

Indirect Rate Distortion Function for Jointly Gaussian RVs

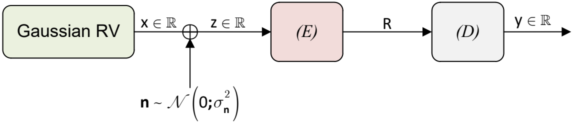


Fig. 8: The quadratic Gaussian remote source coding problem

Gaussian Rate Distortion Problem

Source: $\mathbf{x} \sim \mathcal{N}(0;\sigma_{\mathbf{x}}^2)$, Noisy Measurement: $\mathbf{z} = \mathbf{x} + \mathbf{n}$, $\mathbf{n} \sim \mathcal{N}(0;\sigma_{\mathbf{n}}^2)$ indep. of $\mathbf{x}$

$$R_{\text{in}}(D) = \min_{\mathbf{P}(dz|y):\mathbf{E}\{(\mathbf{x}-\mathbf{y})^2\}\leq D} I(\mathbf{z};\mathbf{y})^* \tag{33}$$

*This problem is studied by many researchers, e.g., [Dobrushin-Tsybakov:1962], [Wolf-Ziv:1970], [Berger:1971]

Remark

Indirect rate distortion problems can be transformed to direct if we modify their distortion constraints [Dobrushin-Tsybakov:1962], [Wolf-Ziv:1970], [Witsenhausen:1980]

Indirect Rate Distortion Function for Jointly Gaussian RVs (cont'd)

Complete Realization and Solution

- $(\mathbf{x},\mathbf{z},\mathbf{y})$ jointly Gaussian
- $\mathbf{P}(dy|z)$ in (33) can be realized as follows

$$\mathbf{y} = H\mathbf{z} + \mathbf{v}, \tag{34}$$

where $H \in \mathbb{R}$ (to be designed), $\mathbf{v} \sim \mathcal{N}(0;\sigma_{\mathbf{v}}^2)$ is an i.i.d Gaussian RV independent of $\mathbf{z}$ with $\sigma_{\mathbf{v}}^2$ (to be designed).

- The design variables $(H,\sigma_{\mathbf{v}}^2)$ are chosen such that: **(i)** $\mathbf{E}\{\mathbf{x}|\mathbf{y}\} = \mathbf{y} - a.s.$; **(ii)** $\mathbf{E}\{(\mathbf{x}-\mathbf{y})^2\} = D$ and are given by

$$H = 1 - \frac{D}{\sigma_{\mathbf{x}}^2},\ \sigma_{\mathbf{v}}^2 = DH - H^2\sigma_{\mathbf{n}}^2. \tag{35}$$

- For D>$D_{\min}$, $R_{\text{in}}(D)$ parametrized by $(H,\sigma_{\mathbf{v}}^2)$ achieves a solution of the form

$$R_{\text{in}}(D) = \frac{1}{2}\left[\log\left(\frac{\sigma_{\mathbf{x}}^2}{D}\right) + \log\left(\frac{\sigma_{\mathbf{x}}^2}{\sigma_{\mathbf{x}}^2 + \sigma_{\mathbf{n}}^2(1-\frac{\sigma_{\mathbf{x}}^2}{D})}\right)\right],\ D \in \left(\text{var}\{\mathbf{x}|\mathbf{z}\}, \sigma_{\mathbf{x}}^2\right], \tag{36}$$

where $\text{var}\{\mathbf{x}|\mathbf{z}\} = \frac{\sigma_{\mathbf{x}}^2\sigma_{\mathbf{n}}^2}{\sigma_{\mathbf{x}}^2+\sigma_{\mathbf{n}}^2}$

Lower bound on $R_{\text{in}}^{\text{op}}(D)$

$$R_{\text{in}}^{\text{na}}(D) = \inf_{\substack{\mathbf{P}(dy_t|y^{t-1},z^t):\ t=0,1\ldots,\infty \\ \limsup_{n\longrightarrow\infty} \frac{1}{n+1}\sum_{t=0}^{n}\mathbf{E}\{||\mathbf{x}_t-\mathbf{y}_t||_2^2\}\le D}} \limsup_{n\longrightarrow\infty} \frac{1}{n+1}\sum_{t=0}^{n} I(\mathbf{z}^t;\mathbf{y}_t|\mathbf{y}^{t-1}) \tag{37}$$

Theorem 1: Time-invariant characterization and realization

- $\{(\mathbf{x}_t,\mathbf{z}_t,\mathbf{y}_t):\ t\in\mathbb{N}_0\}$ jointly Gaussian
- $\mathbf{P}(dy_t|y^{t-1},z^t)$ is conditionally Gaussian with a linear time-invariant Markov realization given by

$$\mathbf{y}_t = H\mathbf{z}_t + (I_p - HC)A\mathbf{y}_{t-1} + \mathbf{v}_t,\ \ \mathbf{v}_t \sim \mathcal{N}(0;\Sigma_{\mathbf{v}}), \Sigma_{\mathbf{v}} \succeq 0 \tag{38}$$

  with design variables $(HC,\Sigma_{\mathbf{v}})$ given by

$$HC = I_p - \Delta\Lambda^{-1},\ \ \Sigma_{\mathbf{v}} = \Delta(HC)^{\mathrm{T}} - H\Sigma_{\mathbf{n}}H^{\mathrm{T}} \succeq 0 \tag{39}$$

- $R_{\text{in}}^{\text{na}}(D)$ parametrized by $(HC,\Sigma_{\mathbf{v}})$ yields the following optimization problem

$$R_{\text{in}}^{\text{na}}(D) = \inf_{\substack{0\prec\Delta\preceq\Lambda \\ 0\prec\Lambda(\Lambda+Q)^{-1}Q\prec\Delta \\ \text{trace}(\Delta)\le D}} \frac{1}{2}\left[\log\frac{\det(\Lambda)}{\det(\Delta)} + \log\frac{\det(\Lambda)}{\det(\Lambda+Q-\Lambda\Delta^{-1}Q)}\right], \tag{40}$$

  where $(\Lambda,\ \Delta)$ are the time-invariant values of $(\Sigma_{t|t-1},\Sigma_{t|t})$ and $Q \triangleq C^{\dagger}\Sigma_{\mathbf{n}}C^{\dagger\,\mathrm{T}} \succeq 0$ with $C^{\dagger} = C^{\mathrm{T}}(CC^{\mathrm{T}})^{-1}$

Lower bound on $R_{\text{in}}^{\text{op}}(D)$ (cont'd)

Technical Comments on Theorem 1

- Proof: **1)** generalized KF recursions for conditionally Gaussian processes where we change the innovations process of the filter to modify the distortion constraint and tranform the problem from partially observable to fully observable; **2)** we use MSE inequalities to make sure that the MMSE estimate $\widehat{\mathbf{x}}_{t|t} = \mathbf{E}\{\mathbf{x}_t|\mathbf{z}^t,\mathbf{y}^t\} = \mathbf{y}_t - a.s.$ i.e., the decoder's output is precisely the optimal linear MMSE estimator and choose accordingly $(HC, \Sigma_{\mathbf{v}})$ so that $\limsup_{n\longrightarrow\infty} \frac{1}{n+1}\sum_{t=0}^{n} \mathbf{E}\{||\mathbf{x}_t - \mathbf{y}_t||_2^2\} = \text{trace}(\Delta)$
- Structural result: The output process $\{\mathbf{y}_t:\ t=0,\ldots\}$ follows a first order Markov process and the directed information measure depends only on the current noisy measurement $\mathbf{z}_t$. Hence (37) simplifies to

$$R_{\text{in}}^{\text{na}}(D) = \inf_{\substack{\mathbf{P}(dy_t|y_{t-1},z_t):\ t=0,1\ldots,\infty\\ \mathbf{E}\{||\mathbf{x}_t-\mathbf{y}_t||_2^2\}\leq D}} \limsup_{n\longrightarrow\infty} \frac{1}{n+1}\sum_{t=0}^{n} I(\mathbf{z}_t;\mathbf{y}_t|\mathbf{y}_{t-1}). \qquad (41)$$

- Special cases: If in (40) we assume that $\Sigma_{\mathbf{n}}=0$, then, $Q=0$ (null matrix) and we recover the characterization for the fully observable case
- The characterization in (40) is a non-convex problem, in general

Sufficient Conditions for Convexification of (40)

Suppose that (Λ, Δ, Q) commute by pairs. Then, (40) simplifies to the following convex program

$$R_{\text{in}}^{\text{na}}(D) = \min \frac{1}{2}\sum_{i=1}^{p}\left[\log\left(\frac{\mu_{\Lambda,i}}{\mu_{\Delta,i}}\right) + \log\left(\frac{\mu_{\Lambda,i}}{\mu_{\Lambda,i}+\mu_{Q,i}-\mu_{\Lambda,i}\mu_{\Delta^{-1},i}\mu_{Q,i}}\right)\right], \tag{42}$$

$$\begin{aligned} \text{s.t.} \qquad & 0 < \frac{\mu_{\Lambda,i}\mu_{Q,i}}{\mu_{\Lambda,i}+\mu_{Q,i}} < \mu_{\Delta,i}, \ \forall i \\ & 0 < \mu_{\Delta,i} \leq \mu_{\Lambda,i}, \ \forall i \\ & \sum_{i=1}^{p}\mu_{\Delta,i} \leq D \end{aligned}$$

where $\mu_{\Lambda,i} = \mu_{A^2,i}\mu_{\Delta,i} + \mu_{\Sigma_{\mathbf{w}},i}$, for some $D \in [D_{\min}, D_{\max}] \subset (0, D_{\max}]$

Proof. The convexity follows because the objective function is differentiable and continuous for $\frac{\mu_{\Lambda,i}\mu_{Q,i}}{\mu_{\Lambda,i}+\mu_{Q,i}} < \mu_{\Delta,i}$. Taking the second partial derivative w.r.t. to $\mu_{\Delta,i}$ it can be shown it is non-negative and the result follows.

Proposition 2: Strong Structural properties for optimality of (42)

Let $(A, Q, \Sigma_{\mathbf{w}})$ satisfy one of the following strong structural properties:

1. A is real symmetric, $\Sigma_{\mathbf{w}} = \sigma_{\mathbf{w}}^2 I_p$ (scalar matrix), where $\sigma_{\mathbf{w}}^2 > 0$, and $Q = qI_p$, (scalar matrix), where $q \geq 0^*$.
2. $A = \alpha I_p$ (scalar matrix), $\Sigma_{\mathbf{w}} = \sigma_{\mathbf{w}}^2 I_p$ (scalar matrix), where $\sigma_{\mathbf{w}}^2 > 0$, and $Q \succeq 0$.
3. $A = \alpha I_p$ (scalar matrix), $\Sigma_{\mathbf{w}} \succ 0$, and $Q = qI_p$ (scalar matrix) where $q \geq 0$.
4. All $(A, \Sigma_{\mathbf{w}}, Q)$ have only diagonal elements with $\Sigma_{\mathbf{w}} \succ 0$ and $Q \succeq 0$;
5. $(A, \Sigma_{\mathbf{w}}, Q)$ have precisely the same matrix structure.
   Then, $(A, Q, \Sigma_{\mathbf{w}}, \Delta)$ commute by pairs and consequently (Δ, Λ, Q) commute

* If $q = 0$, then $Q = 0$ (null matrix).

Reverse-waterfilling solution of (42)

The parametric solution of (42) is

$$R_{\text{in}}^{\text{na}}(D) = \frac{1}{2}\sum_{i=1}^{p}\left[\log\left(\frac{\mu_{\Lambda,i}}{\mu_{\Delta,i}}\right) + \log\left(\frac{\mu_{\Lambda,i}}{\mu_{\Lambda,i}+\mu_{Q,i}-\mu_{\Lambda,i}\mu_{\Delta^{-1},i}\mu_{Q,i}}\right)\right], \tag{43}$$

such that $\mu_{\Lambda,i} = \mu_{A^2,i}\mu_{\Delta,i} + \mu_{\Sigma_{\mathbf{w}},i}$, $\forall i$, and $\mu_{\Delta,i}$ is computed based on the following reverse-waterfilling algorithm

$$\mu_{\Delta,i} = \begin{cases} \xi_i & \text{if } \xi_{\min,i} < \xi_i < \mu_{\Lambda,i} \\ \mu_{\Lambda,i} & \text{if } \xi_i \geq \mu_{\Lambda,i} \end{cases}, \ \forall i, \tag{44}$$

with $\sum_{i=1}^{p}\mu_{\Delta,i} = D$, and $D > D_{\min} = \sum_{i=1}^{p}\xi_{\min,i}$ where

$$\xi_{\min,i} \triangleq \frac{\sqrt{\upsilon^2 + 4\mu_{A^2,i}\mu_{\Sigma_{\mathbf{w}},i}\mu_{Q,i}} - \upsilon}{2\mu_{A^2,i}}, \ \mu_{A,i} \neq 0, \ \forall i, \tag{45}$$

with $\upsilon \triangleq \mu_{\Sigma_{\mathbf{w}},i} + (1-\mu_{A^2,i})\mu_{Q,i}$, $\mu_{Q,i} \neq \infty$, and $\xi_i > \xi_{\min,i}$ is the positive solution of the third degree polynomial equation

$$C1\xi_i^3 + C2\xi_i^2 + C3\xi_i - C4 = 0, \tag{46}$$

where

$$C1 \triangleq 2\mu_{A^4,i}\theta, \qquad C2 \triangleq 2\mu_{A^2,i}\theta(\upsilon + \mu_{\Sigma_{\mathbf{w}},i}),$$

$$C3 \triangleq \mu_{A^2,i}(\upsilon - 2\mu_{\Sigma_{\mathbf{w}},i}) + 2\theta\mu_{\Sigma_{\mathbf{w}},i}(\upsilon - \mu_{A^2,i}\mu_{Q,i}),$$

$$C4 \triangleq \mu_{\Sigma_{\mathbf{w}},i}[2\theta\mu_{\Sigma_{\mathbf{w}},i}\mu_{Q,i} + \mu_{A^2,i}\mu_{Q,i} + \mu_{\Sigma_{\mathbf{w}},i} + \mu_{Q,i}]$$

Technical comments on the Reverse-Waterfilling Solution

- The proof relies on solving KKT conditions. Then, in order to prove that there is exactly one positive solution of the third degree polynomial equation (46) we use Descartes' rule of signs.
- To make sure that the positive solution at each dimension is precisely always $> \xi_{\min,i}$ we need to adjust the global Lagrangian $\theta > 0$

Algorithm 2 Implementation of the reverse-waterfilling solution

Initialize: choose p, ϵ, nominal minimum and maximum value $\theta^{\min}$ and $\theta^{\max}$; choose initial $\mu_{\Lambda,1}$; pick the matrix structure of $(A,\ \Sigma_{\mathbf{w}}, Q)$, and their corresponding eigenvalues $\{(\mu_{A,i},\ \mu_{\Sigma_{\mathbf{w}},i}, \mu_{Q,i}) :\ i \in \mathbb{N}_1^p\}$ (in increasing or decreasing order); Choose $D > D_{\min} = \sum_{i=1}^{p} \xi_{\min,i}$.
Set $\theta = \theta^{\max}$; flag $= 0$.
while flag $= 0$ **do**
  Compute $\mu_{\Delta,i}$ $\forall\ i$ as follows:
  for $i = 1 : p$ **do**
    Compute ξ_i according to (45), (46).
    Compute $\mu_{\Delta,i}$ according to (44).
  end for
  if $\sum_{i=1}^{p} \mu_{\Delta,i} - D \geq \epsilon$ **then**
    Set $\theta^{\min} = \theta$.
  else
    Set $\theta^{\max} = \theta$.
  end if
  if $\theta^{\max} - \theta^{\min} \geq \frac{\epsilon}{p}$ **then**
    Compute $\theta = \frac{(\theta^{\min} + \theta^{\max})}{2}$.
  else
    flag $\leftarrow 1$
  end if
end while
Output: $\{\mu_{\Delta,i} :\ i = 1, 2, \ldots, p\}$, $\{\mu_{\Lambda,i} :\ i = 1, 2, \ldots, p\}$, for a given distortion level D.

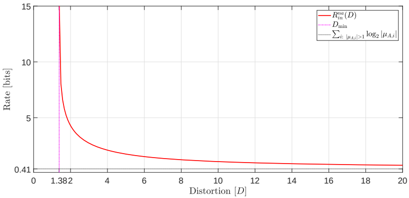
Numerical Simulation

Input data

Consider a time-invariant partially observable Gauss-Markov process with $(A, \Sigma_{\mathbf{w}}, Q)$ given by

$$A = \text{diag}(1.1, 1.1, 1.1.),\ \ \Sigma_{\mathbf{w}} = \text{diag}(1,1,1)\ Q = \begin{bmatrix} 0.4390 & 0.8909 & -0.501 \\ 0.8909 & 1.8145 & -1.0286 \\ -0.501 & -1.0286 & 0.5937 \end{bmatrix}, \tag{47}$$

and $D \in (D_{\min}, 20],\ \ D_{\min} = 1.384.$
$\mu_{Q,1} = 2.836 \geq \mu_{Q,2} = 0.0112 \geq \mu_{Q,1} = 0$ (decreasing order)



- Minimum data rates for MS stability are ≈ 0.41 bits/vector source

Closed-form expression for scalar processes

Closed-form expression: Scalar case*

Consider the scalar-valued version of (29) with
$A \equiv \alpha, \Sigma_{\mathbf{w}} \equiv \sigma^2_{\mathbf{w}}, C \equiv c, \Sigma_{\mathbf{n}} = \sigma^2_{\mathbf{n}}, \Lambda = \lambda, \Delta = D$. Then, its solution is a follows:

$$R^{\text{na}}_{\text{in}}(D) = \frac{1}{2}\left[\log\left(\frac{\lambda}{D}\right) + \log\left(\frac{\lambda}{\lambda + \frac{\sigma^2_{\mathbf{n}}}{c^2}(1 - \frac{\lambda}{D})}\right)\right], \tag{48}$$

where $D \in (D_{\min}, D_{\max}]$ such that

$$D_{\min} = \frac{\sqrt{\upsilon^2 + 4\alpha^2 c^2 \sigma^2_{\mathbf{w}} \sigma^2_{\mathbf{n}}} - \upsilon}{2\alpha^2 c^2}, D_{\max} = \lambda, \tag{49}$$

with $\alpha \neq 0$, $c \neq 0$, $\upsilon \triangleq c^2\sigma^2_{\mathbf{w}} + \sigma^2_{\mathbf{n}}(1 - \alpha^2)$, and $\lambda = \alpha^2 D + \sigma^2_{\mathbf{w}}$.

Proof. Immediate from the solution of KKT conditions.

1. (48) is achieved by a time-invariant realization of the form

$$\mathbf{y}_t = H\mathbf{z}_t + (1 - Hc)\alpha\mathbf{y}_{t-1} + \mathbf{v}_t, \tag{50}$$

where $(Hc, \sigma^2_{\mathbf{v}})$ are given by

$$Hc = 1 - \frac{D}{\lambda},\ \sigma^2_{\mathbf{v}} = DH - H^2\sigma^2_{\mathbf{n}} \tag{51}$$

2. If $\sigma^2_{\mathbf{n}} = 0$, (48) recovers the well-known solution of the fully observable case with $D_{\min} = 0$

Comparison with [Kostina-Hassibi:2019]

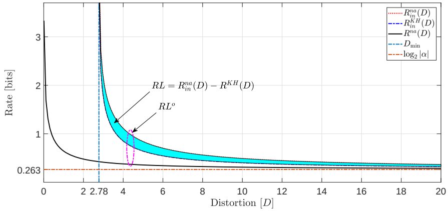
The lower bound proposed in [Kostina-Hassibi:2019]

For $D > D_{\min}^{KH}$,

$$R_{\text{in}}^{KH}(D) = \frac{1}{2}\log\left(\frac{\lambda - D_{\min}^{KH}}{D - D_{\min}^{KH}}\right) \quad (52)$$

with $D_{\max} = \lambda = \alpha^2 D + \sigma_{\mathbf{w}}^2$; $D_{\min}^{KH} = \text{var}\{\mathbf{x}_t|\mathbf{z}^t\}$ obtained via the steady-state of a pre-KF algorithm (it can be shown that $D_{\min}^{KH} \equiv D_{\min}$)

Example. Input data $\alpha = 1.2$, $c = 0.4$, $\sigma_{\mathbf{w}}^2 = \sigma_{\mathbf{n}}^2 = 1$, $D \in (D_{\min}, 20]$ (remote case) $D \in (0, 20]$ (fully observable case)



- $RL \approx 0.42$ bits (for this example); it can be less or more depending on the input data
- $RL^o = \frac{1}{2}\log\left(\frac{\lambda}{\lambda + \frac{\sigma_{\mathbf{n}}^2}{c^2}(1 - \frac{\lambda}{D})}\right)$, $D \in (D_{\min}, D_{\max}]$,
- Minimum data rates for MS stability are ≈ 0.263 bits/sample

Upper bound on $R_{\text{in}}^{\text{op}}(D)$

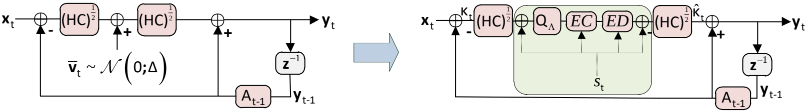


Fig. 9: Optimal realization that achieves the lower bound $R_{\text{in}}^{\text{na}}(D)$ (left figure) using structural properties and the same realization with the additive Gaussian noise replaced by coding noise via ECDQ scheme (right figure)

1. The realization that achieves $R_{\text{in}}^{\text{na}}(D)$ can be equivalently written as

$$\mathbf{y}_t = HC(\mathbf{x}_t - A\mathbf{y}_{t-1}) + A\mathbf{y}_{t-1} + \bar{\mathbf{v}}_t, \tag{53}$$

where $\bar{\mathbf{v}}_t = H\mathbf{n}_t + \mathbf{v}_t \sim \mathcal{N}(0; \Delta(HC))$

2. The coding scheme follows precisely like the fully observable case (different scalings)
3. Upper bound:

$$R_{\text{in}}^{\text{op}}(D) \leq R_{\text{in}}^{\text{na}}(D) + \frac{r}{2}\log(2\pi e G_r) \tag{54}$$

with $r = \text{rank}(HC)$

Open Question: Bounds for LQG Control

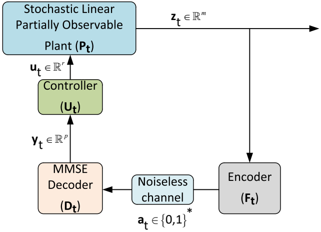


Fig. 10: NCS with a single-loop

Operational Rate Cost Function

$$R_{\text{in}}^{\text{op}}(\Gamma) = \inf_{\substack{(\mathbf{E}_t, \mathbf{D}_t, \mathbf{U}_t):\ t=0,\ldots,\infty \\ \limsup_{n\longrightarrow\infty} \frac{1}{n+1} LQR(\mathbf{x}^n, \mathbf{u}^{n-1}) \leq \Gamma}} \limsup_{n\longrightarrow\infty} \frac{1}{n+1} \sum_{t=0}^{n} H(\mathbf{y}_t | \mathbf{y}^{t-1}, \mathbf{u}^{t-1}) \quad (55)$$

where $LQR(\mathbf{x}^n, \mathbf{u}^{n-1}) \triangleq \mathbf{E}\left[\sum_{t=0}^{n-1} \{||\mathbf{x}_t||_Q^2 + ||\mathbf{u}_t||_R^2\} + ||\mathbf{x}_n||_Q^2\right]$

Questions:

1. Bounds on (55) via optimal reverse-waterfilling solutions
2. Going beyond additive Gaussian processes

Thank you!

QUESTIONS

For more information:

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