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On the tightness of linear policies for stabilization of linear systems over Gaussian networks

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ABSTRACT

In this paper, we consider stabilization of multi-dimensional linear systems driven by Gaussian noise controlled over parallel Gaussian channels. For such systems, it has been recognized that for stabilization in the sense of asymptotic stationarity or stability in probability, Shannon capacity of a channel is an appropriate measure on characterizing whether a system can be made stable when controlled over the channel. However, this is in general not the case for quadratic stabilization. On a related problem of joint-source channel coding, in the information theory literature, the source-channel matching principle has been shown to lead to optimality of uncoded or analog transmission and when such matching conditions occur, it has been shown that capacity is also a relevant figure of merit for quadratic stabilization. A special case of this result is applicable to a scalar LQG system controlled over a scalar Gaussian channel. In this paper, we show that even in the absence of source-channel matching, to achieve quadratic stability, it may suffice that information capacity (in Shannon's sense) is greater than the sum of the logarithm of unstable eigenvalue magnitudes. In particular, we show that periodic linear time varying coding policies are optimal in the sense of obtaining a finite second moment for the state of the system with minimum transmit power requirements for a large class of vector Gaussian channels. Our findings also extend the literature which has considered noise-free systems.

1. Problem formulation

Consider the following linear time invariant system:

$$\bar{X}_{t+1} = A\bar{X}_t + B\bar{U}_t + \bar{W}_t, \quad t \in \mathbb{N}, \tag{1}$$

where $\bar{X}_t \in \mathbb{R}^n$ is a state process, $\bar{U}_t \in \mathbb{R}^n$ is a control process, $\bar{W}_t \in \mathbb{R}^n$ is an independent and identically distributed sequence of Gaussian random variables with zero mean and covariance K_W. The system matrix A and the input matrix B are of appropriate dimensions and we assume that the pair (A, B) is controllable. Let $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be the eigenvalues of the system matrix A. Without loss of generality we assume that all the eigenvalues of A are outside the unit disc ($1 \leq |\lambda_i| < \infty$ for all i), i.e., all modes are unstable. The initial state of the system X_0 is assumed to be a random variable with zero mean and covariance Λ_0 with Trace $\{\Lambda_0\} < \infty$. The initial state X_0 is assumed to be independent of the plant noise variable $\bar{W}_t$. Consider the scenario depicted in Fig. 1, where a sensor observes an n-dimensional state process and transmits it to a remote controller over m parallel independent Gaussian channels. At any time instant t, $S_t := [s_{1,t}, s_{2,t}, \ldots, s_{m,t}]$ and $R_t := [r_{1,t}, r_{2,t}, \ldots, r_{m,t}]$ are the input and output of the channel, where $r_{i,t} = s_{i,t} + z_{i,t}$ and $z_{i,t} \sim \mathcal{N}(0, N_i)$ are zero mean white Gaussian noise components with $N_1 \leq N_2 \leq \cdots \leq N_m$. We assume that there is a noiseless causal feedback link from the controller to the sensor and the plant. Let $f_t : \mathbb{R}^{(n+m)t+n} \to \mathbb{R}^m$ denote the sensing policy such that $S_t = f_t(X_{[0,t]}, R_{[0,t-1]})$, where $X_{[0,t]} := \{X_0, X_1, \ldots, X_t\}$ and the sensor is assumed to have an average transmit power constraint $\mathbb{E}[\|S_t\|^2] \leq P_S$. Further, let $\pi_t : \mathbb{R}^{m(t+1)} \to \mathbb{R}^n$ be the controller policy, then we have $U_t = \pi_t(R_{[0,t]})$. The common goal of the sensor and the controller is to stabilize the system (1) in the mean square sense, defined as follows.

Definition 1.1 (*[1, Definition 2.2]*)**.** A system is said to be *mean square stable* if there exists $M < \infty$ such that $\sup_t \mathbb{E}[\|X_t\|^2] < M$.

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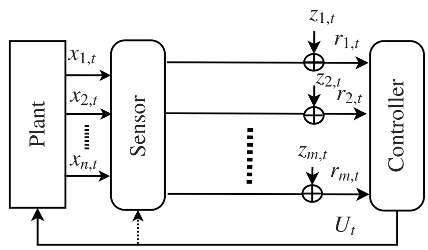

Fig. 1. Control over parallel Gaussian channels.

Literature review. Stabilization of linear systems over communication channels has been studied in [2–4,1,5–17]. If the goal is stabilization in the sense of asymptotic mean stationarity [18] or similar notions such as stability in probability [1,6], Shannon capacity is the right measure on what is possible or not (see [18] for a detailed account of such results in the literature). But if the goal is stabilization in the sense of having finite second moments, then Shannon capacity may not the right measure [1].

A related, but different problem, is the joint source-channel coding problem. In this context, the Gaussian source-channel setting is an important special case due to the possible mean-square optimality of linear coding policies as a consequence of what is known as source-channel matching [19–21]. Using the data-processing theorem of information theory and dynamic programming (see e.g. [20,22,18]), it can be shown that when source-channel matching occurs, linear coding policies for controlled linear Gaussian sources are optimal for the minimization of quadratic distortion measures across Gaussian channels. However, such a source-channel matching does not apply in a large class of settings involving multi-dimensional sources and channels [23–26]. Along a similar context, we note that recent work [27] has obtained structural properties of channels which can be used to realize optimal causal channel codes for a class of multi-dimensional Gaussian sources with memory.

In the control literature, these problems have been considered where the sufficiency of information capacity[1] being greater than a lower bound has been observed in a class of settings [10–12], which, however, consider *noise-free plants*. It is observed in [11] that LTI schemes are not optimal for stabilization over parallel channels. For optimal encoding in the unmatched case, linear encoding is also not optimal in general [29,30]. Even in the class of memoryless coding schemes, linear coding is not optimal for the transmission of memoryless Gaussian sources over memoryless Gaussian channels with quadratic distortion measures [23,25,31,32].

That a scalar Gaussian channel allows for stability when its information capacity is greater than the sum of the logarithm of unstable eigenvalue magnitudes of a linear system, not only in the sense of ergodicity but also in the sense of quadratic stability, is not surprising. The reason for this argument is that for such channels, the data processing inequality arguments lead to the optimality of linear coding and decoding policies for the minimization of the quadratic estimation error for the state (see *Chapter 11* in [18]). One can also show that for a scalar Gaussian channel, the error exponent with feedback is not bounded [33–35] and using arguments in [1,14,18], one expects that quadratic stability is possible even for systems driven by unbounded noise.

Results on controlling a vector linear system over a scalar Gaussian channel have been obtained in [36] confirming this line of thought, where linear time-varying policies have been shown to be sufficient for mean-square stability. However, there does not exist result in the literature that considers noisy multi-dimensional linear systems controlled over multi-dimensional Gaussian channels. For such channels, in general, the information theoretic approach based on the data-processing inequality does not lead to tight bounds on optimal joint-source-channel coding schemes, unlike the scalar case.

Contributions of the paper: In this paper, we consider quadratic (second moment) stabilization of multi-dimensional linear systems (sources) represented by (1) over vector-valued Gaussian channels. We show that for a large class of source-channel pairs, information capacity being greater than the sum of the logarithm of unstable eigenvalue magnitudes of the linear system (1) is sufficient for quadratic stability and linear sensing and control schemes are optimal, even when the source-channel matching principle does not hold.

In the literature, stabilization results have been presented for *noiseless* multi-dimensional plants over multidimensional channels in [11,12,36,37] and for *noisy* multi-dimensional plants over *scalar* channels in [36]. Our paper extends these results to more general setups and establishes optimality of linear sensing and control schemes for the moment stabilization of a wide class of *noisy* linear plants over vector Gaussian channels.[2]

2. Sufficient conditions and a linear time-varying scheme

We have the following sufficiency result.

Theorem 2.1. *The system (1) can be mean square stabilized over m parallel independent Gaussian channels using a linear scheme if there exist* $f_{ij} \in \mathbb{Q}$ *such that* $f_{ij} \geq 0$, $\sum_{j=1}^{m} f_{ij} \leq 1$, $\sum_{i=1}^{n} f_{ij} \leq 1$ *and*

$$\log(|\lambda_i|) < \sum_{j=1}^{m} f_{ij} C_j, \quad \forall i \in \{1, 2, \ldots, n\}, \tag{2}$$

where λ_i *are eigenvalues of the system matrix A in (1) and* $C_j := \frac{1}{2}\log(1+\frac{P_j}{N_j})$ *is the information capacity of jth channel.*

Proof. For the proof, we propose a periodic linear time varying scheme sensing and control scheme. We first give the scheme for a system with invertible input matrix B, assuming that $B = I$ in (1): Consider that the control actions in (1) are taken periodically after every K time steps, i.e., at $t = lK - 1$ for $l \in \mathbb{N}$ ($U_t = 0$ for $t \neq lK - 1$). Under this control strategy, the state equation at $t = lK$ is given by

$$\bar{X}_{t+K} = A^K \bar{X}_t + \bar{U}_{t+K-1} + \sum_{i=0}^{K-1} A^{K-i-1} \bar{W}_{t+i}. \tag{3}$$

For $A^K \in \mathbb{R}^{n\times n}$ there exist a real non-singular matrix T and a real matrix $\tilde{A}$ such that $\tilde{A} = T^{-1}A^K T = \mathrm{diag}\left[J_1, \ldots, J_p\right]$, where J_p is a Jordan block of dimension (algebraic multiplicity) n_p [39]. A Jordan block $J_p \in \mathbb{R}^{n_p\times n_p}$ associated with a real eigenvalue λ of multiplicity n_p has the following form:

$$J_p = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}, \tag{4}$$

[1] The definition of information capacity for Gaussian channels can be found in page 263 in [28].

[2] Part of results without proofs have been included in a book chapter that provides an overview of some recent results on stabilization and control over Gaussian networks [38].

and a Jordan block $J_p \in \mathbb{R}^{n_p \times n_p}$ associated with a complex conjugate pair of eigenvalues $\lambda = \sigma \pm j\omega$ is given by,

$$J_p = \begin{pmatrix} D & I & & \\ & D & \ddots & \\ & & \ddots & I \\ & & & D \end{pmatrix} \tag{5}$$

where $D = \begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}$. Now apply the linear transformation $X_t = T^{-1}\bar{X}_t$ such that $T^{-1}A^K T$ is in a real Jordan normal form. Under this transformation, (3) is written as

$$X_{t+K} = \tilde{A}X_t + U_{t+K-1} + V_t, \quad \text{for } t = lK,\ l \in \mathbb{N}, \tag{6}$$

where $\tilde{A} := T^{-1}A^K T$, $U_t := T^{-1}\bar{U}_t$, and $V_t := T^{-1}\sum_{i=0}^{K-1} A^{K-i-1} \bar{W}_{t+i}$. The matrix $\tilde{A}$ is in real Jordan form with eigenvalues $\tilde{\lambda}_i = \lambda_i^K$, where λ_i are the eigenvalues of A.

Now consider that the sensor observes the state vector $X_t := [x_{1,t}, x_{2,t}, \ldots, x_{n,t}]^T$ periodically at $t = lK$. The sensor has access to m parallel Gaussian channels over which it wishes to communicate the state vector X_t to the remote decoder/controller. We propose the following periodic linear transmit strategy: Consider that within each period of K time steps, the encoder linearly transmits different components $x_{i,t}$ of the state vector X_t on different channels such that the following two conditions are satisfied in every time step t: (i) Each channel is used for the transmission of at most one state component, i.e., two state components are not transmitted over one channel simultaneously, (ii) None of the state components is transmitted over more than one channel simultaneously. Let $k_{ij} \in \mathbb{N}$ be the number of times the jth channel having information capacity $C_j = \frac{1}{2}\log(1 + \frac{P_j}{N_j})$ is used to transmit the state $x_{i,t}$. Under the proposed scheme, we have

$$k_{ij} \geq 0, \qquad \sum_{j=1}^{m} k_{ij} \leq K, \qquad \sum_{i=1}^{n} k_{ij} \leq K. \tag{7}$$

We assume that $x_{i,t}$ is Gaussian distributed due to the following argument: If the initial state $x_{i,0}$ is not Gaussian distributed, then one can perform an initialization step as in [36, Appendix B, page 2379] to make it Gaussian. This Gaussianization step was first introduced in [34] for the problem of reliable communication over a Gaussian channel with noiseless feedback and it has been used in [36,40,41] for the problem of stabilization over Gaussian channels with noiseless feedback. After performing this initialization step, the state is always Gaussian distributed since the sensing and control policies are linear and the noise variables are Gaussian. Therefore without loss of generality, we assume that $x_{i,t}$ is Gaussian distributed.

Let $\hat{x}_{i,t}$ denote the decoder's MMSE estimate of $x_{i,t}$ at the end of each transmission period of K time steps. It is shown in Appendix A that under the proposed linear scheme, minimum mean-squared error of each state component is given by

$$\mathbb{E}\left[\left(x_{i,t} - \hat{x}_{i,t}\right)^2\right] = 2^{-2\sum_{j=1}^{m} k_{ij}C_j}\, \mathbb{E}\left[x_{i,t}^2\right]. \tag{8}$$

Let us define $\hat{X}_t := [\hat{x}_{1,t}, \hat{x}_{2,t}, \ldots, \hat{x}_{n,t}]^T$. The controller then takes following actions $U_{t+K-1} = -\tilde{A}\hat{X}_t$ at $t = lK - 1$. Let $f_{ij} := \frac{k_{ij}}{K}$. It is shown in Appendix B that if we can choose $\{K, k_{ij}\}$ such that f_{ij} satisfies the conditions given in Theorem 2.1, then the plant is mean-square stable.

Theorem 2.1 holds for a system with a controllable (A, B) pair by the following argument: For a system with controllable (A, B), any input can be realized by n consecutive actions of the controller. Since the state encoder has access to the channel outputs, the encoder–decoder pair can always keep refining estimates of the state components during the K time steps according to the scheme given in Appendix A. At the end of K time steps, an estimate $\hat{X}$ is available at the controller side. Now the controller wishes to apply an input $-A\hat{X}$, which can be realized in the following n time steps due to the assumption that the pair (A, B) is controllable. During these n time steps when the controller is applying actions to realize the input $-A\hat{X}$, the encoder–decoder pair will keep refining the state estimate. □

3. Tightness of the linear scheme

We first present a necessary condition for stabilization.

Theorem 3.1. *The linear system in (1) can be mean square stabilized over the given parallel Gaussian channel only if*

$$\log\left(|\det(A)|\right) < \frac{1}{2}\sum_{j=1}^{m}\log\left(1 + \frac{P_j^\star}{N_j}\right), \tag{9}$$

where $P_j^\star = \max\{\gamma - N_j, 0\}$ and γ is chosen such that $\sum_{j=1}^{m} P_j^\star = P_S$.

Proof. The proof simply follows from [14, Theorem 4.1] and the fact that the R.H.S. of the inequality in (9) is the information capacity of the parallel Gaussian channel [42]. Note that the proof in [14] applies for the Gaussian channel by approximating it as a limit of discrete channels [43, Lemma 5.5.1]. □

Papers [11,12] derive conditions for mean-square stabilization of *noiseless* linear plants over parallel Gaussian channels. The necessary condition in [11, Theorem 6] is not tight in general and its achievability is not guaranteed by LTI schemes. The paper [12] proposes a non-linear scheme for a *noise-free scalar* plant and derives a sufficient condition [12, Theorem 6], which coincides with the necessary condition (9). Thus the non-linear scheme in [12] is optimal for stabilization of *noiseless scalar* plant. A time varying version of this non-linear scheme is optimal for stabilization of *noiseless multi-dimensional* plants [36]. In the following we present the conditions under which the proposed linear scheme is optimal for mean-square stabilization of *noisy multi-dimensional* plants over *vector* Gaussian channels.

Theorem 3.2. *The linear scheme is optimal for mean-square stabilizing an n-dimensional plant over m parallel Gaussian channels if there exist $f_{ij} \in \mathbb{Q}$ such that $f_{ij} \geq 0$, $\sum_{j=1}^{m^\star} f_{ij} \leq 1$, $\sum_{i=1}^{n} f_{ij} = 1$ and*

$$\log\left(|\lambda_i|\right) < \sum_{j=1}^{m^\star}\frac{f_{ij}}{2}\log\left(1 + \frac{P_j^\star}{N_j}\right), \tag{10}$$

for all $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m^\star\}$, where $P_j^\star$ is the optimal power allocation given by the water-filling solution [42, pp. 204–205] and $m^\star \leq m$ is the number of active channels for which optimal transmit power is non-zero.

Proof. According to the water-filling solution [42, pp. 204–205], the optimal power allocation over sub-channels for maximizing capacity is given by $P_j^\star = \max\{\gamma - N_j, 0\}$, where γ is chosen such that $\sum_{i=j}^{m} P_j^\star = P_S$. Since we have assumed $N_1 \leq N_2 \leq \cdots \leq N_m$, there exists $0 \leq m^\star \leq m$ such that $P_j^\star > 0$ for $j \leq m^\star$ and $P_j^\star = 0$ for $j > m^\star$. Suppose we allocate the powers to the sub-channels according to the water filling solution, i.e., we now have $m^\star$ active parallel channels with capacities $C_j = \frac{1}{2}\log\left(1 + \frac{P_j^\star}{N_j}\right)$ for

$1 \le j \le m^\star$. According to Theorem 2.1, the system in (1) is mean-square stable under the linear time varying scheme if there exist $f_{ij} \in \mathbb{Q}$ such that $f_{ij} \ge 0$, $\sum_{j=1}^{m} f_{ij} \le 1$, $\sum_{i=1}^{n} f_{ij} \le 1$ and

$$\log(|\lambda_i|) < \sum_{j=1}^{m^\star} \frac{f_{ij}}{2} \log\left(1 + \frac{P_j^\star}{N_j}\right), \tag{11}$$

for all $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m^\star\}$. Summing over $1 \le i \le n$, we get

$$\sum_{i=1}^{n} \log(|\lambda_i|) < \sum_{i=1}^{n}\sum_{j=1}^{m} f_{ij} C_{ij} = \sum_{j=1}^{m}\left(\sum_{i=1}^{n} f_{ij}\right) C_j. \tag{12}$$

Observe that if $\sum_{i=1}^{n} f_{ij} = 1$, then (12) can be written as

$$\log(|\det(A)|) = \sum_{i=1}^{n} \log(|\lambda_i|) < \frac{1}{2}\sum_{j=1}^{m} \log\left(1 + \frac{P_j^\star}{N_j}\right), \tag{13}$$

which is the same condition as (9), i.e., the necessary and the sufficient conditions coincide. Thus the proposed scheme is optimal in the sense that there does not exist any other scheme that can stabilize the plant with a lower transmission power, when there exist coefficients f_{ij} that satisfy the conditions given in Theorem 3.2. □

Remark 3.1. A general vector Gaussian channel can be decomposed into an equivalent parallel channel by employing linear pre-processing at the encoder and linear post-processing at the decoder [42, pp. 292]. If the equivalent parallel channel satisfies the conditions in Theorem 3.2, then the proposed linear scheme is also optimal over the given vector Gaussian channel.

Remark 3.2. The verification of the existence of coefficients f_{ij} satisfying the conditions given in Theorem 3.2 is a linear program and is computationally feasible. In the following we provide some particular instances where linear scheme is optimal. In all the following examples, we assume that $\sum_{i=1}^{n} \log(|\lambda_i|) < \sum_{j=1}^{m} C_j$ because this is a necessary condition for stabilization.

1. If $n = m$, $\log(\lambda_i) < C_i$, then we can choose $f_{ii} = 1$ and $f_{ij} = 0$ for $j \neq i$.
2. If $m = 2$, $n = 3$, with $\lambda_1 = \lambda_2 = \lambda_3^l$, $C_2 = \frac{l}{2} C_1$ for any $l \in \mathbb{N}$, then we can choose, $f_{11} = f_{21} = 0.5$, $f_{31} = 0$, $f_{12} = f_{22} = 0$, $f_{32} = 1$.
3. If $m = 1$ (scalar channel), then we can choose $f_{i1} = \frac{\log(|\lambda_i|)}{\sum_{i=1}^{n}\log(|\lambda_i|)}$ for all i.

4. Conclusions

We studied the problem of mean-square stabilization of a *noisy multi-dimensional* linear systems over *vector* Gaussian channels subject to an average transmit power constraint. A linear time varying sensing and control scheme is proposed and the conditions which guarantee optimality of the linear scheme are derived. For a given system and a given channel, the optimality of the linear policies can be verified by solving a linear program. We observe that the linear scheme is optimal for a wide class of linear systems and Gaussian channels in the sense that there does not exist any other scheme that can mean-square stabilize the system using a lower transmission power. Interestingly, linear policies are optimal for quadratic stabilization even if the source-channel matching principle does not hold.

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Appendix A. Estimation with noiseless feedback

For simplification we drop the subscripts of $x_{i,t}$ and consider that a variable x has to be transmitted/estimated over m parallel Gaussian channels, where jth sub-channel has capacity $C_j = \frac{1}{2}\log(1 + P_j/N_j)$ for $j = 1, 2, \ldots, m$. Assume that the encoder uses the first channel k_1 times, the second channel k_2 times, and so on. Consider the following scheme based on the Schalkwijk Kailath coding scheme [34]: At $t = 1$, the encoder transmits $s_{1,1} = \sqrt{\frac{P_1}{\mathbb{E}[x^2]}}x$ over the first channel, the decoder receives $r_{1,1} = s_{1,1} + z_{1,1}$, and computes the MMSE estimate $\hat{x}_1 = \frac{\mathbb{E}[xr_{1,1}]}{\mathbb{E}[r_{1,1}^2]} r_{1,1}$, where $\hat{x}_t$ denotes the estimate of x at time t. Further the estimation error at any time t is denoted as $\epsilon_t := x - \hat{x}_t$ and the encoder can compute the error due to noiseless feedback link. By computation, the variance of ϵ_1 is

$$\mathbb{E}[\epsilon_1^2] = \frac{N_1}{P_1 + N_1}\mathbb{E}[x^2] = 2^{-2C_1}\mathbb{E}[x^2], \tag{A.1}$$

where $C_1 = \frac{1}{2}\log(1 + P_1/N_1)$. For $2 \le t \le k_1$, the encoder transmits $s_{1,t} = \sqrt{\frac{P_1}{\mathbb{E}[\epsilon_{t-1}^2]}}\epsilon_{t\text{-}1}$, the decoder estimates $\hat{\epsilon}_{t\text{-}1} = \frac{\mathbb{E}[\epsilon_{t-1} r_{1,t}]}{\mathbb{E}[r_{1,t}^2]} r_{1,t}$ and updates its estimate of x as $\hat{x}_t = \hat{x}_{t\text{-}1} - \hat{\epsilon}_{t\text{-}1}$. The associated estimation error is, $\epsilon_t = x - \hat{x}_t = \hat{\epsilon}_{t\text{-}1} - \epsilon_{t\text{-}1}$. The variance of estimation error is computed as $\mathbb{E}[\epsilon_t^2] = 2^{-2C_1}\mathbb{E}[\epsilon_{t-1}^2]$, which together with (A.1) implies

$$\mathbb{E}[\epsilon_{k_1}^2] = 2^{-2k_1 C_1}\mathbb{E}[x^2]. \tag{A.2}$$

Similarly, for the next k_2 time steps the encoder transmits over the second sub-channel having capacity C_2, and then over the third channel and so on. At the end of transmission over the jth sub-channel the variance of estimation error is given by

$$\mathbb{E}[\epsilon_{k_1+k_2+\cdots+k_j}^2] = 2^{-2k_j C_j}\mathbb{E}[\epsilon_{k_{j\text{-}1}}^2]. \tag{A.3}$$

Accordingly the estimation error at the end of whole transmission period, i.e., at $t = \sum_{j=1}^{m} k_j$, is given by

$$\mathbb{E}[(x - \hat{x}_t)^2] = 2^{-2k_m C_m}\mathbb{E}[\epsilon_{k_{m\text{-}1}}^2] = 2^{-2\sum_{j=1}^{m} k_j C_j}\mathbb{E}[x^2]. \tag{A.4}$$

Appendix B. Proof of Theorem 2.1

Under the proposed linear scheme, we can write (6) as

$$X_{t+K} = \tilde{A}\left(X_t - \hat{X}_t\right) + V_t, \quad t = lK, l \in \mathbb{N}, \tag{B.1}$$

where $\tilde{A}$ is in real Jordan form with eigenvalues $\tilde{\lambda}_i$. Some of these eigenvalues can be either real or complex and distinct or have algebraic multiplicity. According to (B.1), the state component corresponding to a real distinct eigenvalue $\tilde{\lambda}_i$ is given by

$$x_{i,t+K} = \tilde{\lambda}_i(x_{i,t} - \hat{x}_{i,t}) + v_{i,t}. \tag{B.2}$$

For a complex eigenvalue pair $\tilde{\lambda}_i = \tilde{\lambda}_{i+1}^*$, the state components are given by

$$\begin{aligned} x_{i,t+K} &= \tilde{\sigma}(x_{i,t} - \hat{x}_{i,t}) + \tilde{\omega}(x_{i+1,t} - \hat{x}_{i+1,t}) + v_{i,t}, \\ x_{i+1,t+K} &= -\tilde{\omega}(x_{i,t} - \hat{x}_{i,t}) + \tilde{\sigma}(x_{i+1,t} - \hat{x}_{i+1,t}) + v_{i+1,t}. \end{aligned} \tag{B.3}$$

If $\tilde{\lambda}_r = \tilde{\lambda}_{r+1} = \cdots = \tilde{\lambda}_s = \tilde{\lambda}$ for some $r \le s$, then according to (4) and (B.1) the corresponding states are given by

$$\begin{aligned} x_{i,t+K} &= \tilde{\lambda}(x_{i,t} - \hat{x}_{i,t}) + (x_{i+1,t} - \hat{x}_{i+1,t}) + v_{i,t}, \\ x_{s,t+K} &= \tilde{\lambda}(x_{s,t} - \hat{x}_{s,t}) + v_{s,t}, \end{aligned} \tag{B.4}$$

for $i = r, \ldots, s-1$. Finally, if $\tilde{\lambda}_r = \tilde{\lambda}^*_{r+1} = \tilde{\lambda}_{r+2} = \tilde{\lambda}^*_{r+3} \cdots = \tilde{\lambda}^\star_s = \tilde{\sigma} + j\tilde{\omega}$ for some $r \le s$, then according to (5) and (B.1) the corresponding states are given by

$$\begin{aligned} x_{i,t+K} &= \tilde{\sigma}(x_{i,t} - \hat{x}_{i,t}) + \tilde{\omega}(x_{i+1,t} - \hat{x}_{i+1,t}) + (x_{i+2,t} - \hat{x}_{i+2,t}) \\ &\quad + v_{i,t}, \quad \text{for } i = r, r+2, \ldots, s-2, \\ x_{i+1,t+K} &= -\tilde{\omega}(x_{i,t} - \hat{x}_{i,t}) + \tilde{\sigma}(x_{i+1,t} - \hat{x}_{i+1,t}) \\ &\quad + (x_{i+3,t} - \hat{x}_{i+3,t}) + v_{i+1,t}, \\ &\quad \text{for } i = r+1, r+3, \ldots, s-1, \\ x_{s-1,t+K} &= \tilde{\sigma}(x_{s\text{-}1,t} - \hat{x}_{s\text{-}1,t}) + \tilde{\omega}(x_{s,t} - \hat{x}_{s,t}) + v_{s\text{-}1,t}, \\ x_{s,t+K} &= -\tilde{\omega}(x_{s\text{-}1,t} - \hat{x}_{s\text{-}1,t}) + \tilde{\sigma}(x_{s,t} - \hat{x}_{s,t}) + v_{s,t}. \end{aligned} \tag{B.5}$$

In the following we find conditions which are sufficient for mean-square stabilization of the state components given by (B.2), (B.3), (B.4), and (B.5), that covers all possible Jordan blocks for the system matrix A. We first consider the state equation (B.2) corresponding to a unique real eigenvalue $\tilde{\lambda}_i$. Let k_{ij} be the number of times the jth channel is used for the transmission of state $x_{i,t}$ associated with the unique real eigenvalue $\tilde{\lambda}_i$. The second moment of $x_{i,t+K}$ at $t = lK$ is,

$$\begin{aligned} \mathbb{E}\left[x^2_{i,t+K}\right] &= \tilde{\lambda}_i^2 \mathbb{E}\left[(x_{i,t} - \hat{x}_{i,t})^2\right] + \mathbb{E}\left[v^2_{i,t}\right] \\ &\overset{(a)}{=} \tilde{\lambda}_i^2 2^{-2\sum_{j=1}^m k_{ij}C_j} \mathbb{E}\left[x^2_{i,t}\right] + \nu_i, \end{aligned} \tag{B.6}$$

where (a) follows from (8) and $\nu_i := \mathbb{E}\left[v^2_{i,t}\right]$ for $i = 1, 2, \ldots, n$. We observe that $\mathbb{E}\left[x^2_{i,t}\right]$ is bounded if $\tilde{\lambda}_i^2 2^{-2\sum_{j=1}^m k_{ij}C_j} < 1$. Thus the state $x_{i,t}$ is stable if

$$\tilde{\lambda}_i^2 2^{-2\sum_{j=1}^m k_{ij}C_j} < 1 \Rightarrow \log\left(|\tilde{\lambda}_i|\right) < \sum_{j=1}^m k_{ij}C_j. \tag{B.7}$$

Next consider the states in (B.3), which are associated with a unique complex eigenvalue pair $\tilde{\lambda}_i, \tilde{\lambda}_{i+1}$. Since $|\tilde{\lambda}_i| = |\tilde{\lambda}_{i+1}|$, we assume that $k_{ij} = k_{i+1j}$, i.e., all the channels are equally used for the transmission of $x_{i,t}$ and $x_{i+1,t}$. The second moments of $x_{i,t+K}$ and $x_{i+1,t+K}$ at $t = lK$ are given by

$$\begin{aligned} \mathbb{E}\left[x^2_{i,t+K}\right] &= 2^{-2\sum_{j=1}^m k_{ij}C_j} \left(\tilde{\sigma}^2 \mathbb{E}\left[x^2_{i,t}\right] + \tilde{\omega}^2 \mathbb{E}\left[x^2_{i+1,t}\right]\right) \\ &\quad + 2\tilde{\sigma}\tilde{\omega}\mathbb{E}\left[\left(x_{i,t} - \hat{x}_{i,t}\right)\left(x_{i+1,t} - \hat{x}_{i+1,t}\right)\right] + \nu_i, \end{aligned} \tag{B.8}$$

$$\begin{aligned} \mathbb{E}\left[x^2_{i+1,t+K}\right] &= 2^{-2\sum_{j=1}^m k_{ij}C_j} \left(\tilde{\omega}^2 \mathbb{E}\left[x^2_{i,t}\right] + \tilde{\sigma}^2 \mathbb{E}\left[x^2_{i+1,t}\right]\right) \\ &\quad - 2\tilde{\sigma}\tilde{\omega}\mathbb{E}\left[\left(x_{i,t} - \hat{x}_{i,t}\right)\left(x_{i+1,t} - \hat{x}_{i+1,t}\right)\right] + \nu_{i+1}. \end{aligned} \tag{B.9}$$

By summing (B.8) and (B.9) we get,

$$\begin{aligned} \mathbb{E}\left[x^2_{i,t+K}\right] + \mathbb{E}\left[x^2_{i+1,t+K}\right] &= \left(\tilde{\sigma}^2 + \tilde{\omega}^2\right) 2^{-2\sum_{j=1}^m k_{ij}C_j} \\ &\quad \times \left(\mathbb{E}\left[x^2_{i,t}\right] + \mathbb{E}\left[x^2_{i+1,t}\right]\right) + \nu_i + \nu_{i+1}. \end{aligned} \tag{B.10}$$

The sum $\mathbb{E}\left[x^2_{i,t+K}\right] + \mathbb{E}\left[x^2_{i+1,t+K}\right]$ is bounded if $\left(\tilde{\sigma}^2 + \tilde{\omega}^2\right) 2^{-2\sum_{j=1}^m k_{ij}C_j} < 1$. Since $|\tilde{\lambda}_i|^2 = |\tilde{\lambda}_{i+1}|^2 = \left(\tilde{\sigma}^2 + \tilde{\omega}^2\right)^2$ and $k_{ij} = k_{i+1j}$, we have the following conditions for stabilization:

$$\log\left(|\tilde{\lambda}_i|\right) < \sum_{j=1}^m k_{ij}C_j, \quad \log\left(|\tilde{\lambda}_{i+1}|\right) < \sum_{j=1}^m k_{i+1j}C_j. \tag{B.11}$$

Now consider the state components $\{x_{i,t}\}_{i=r}^s$ corresponding to the Jordan block associated with real eigenvalue given in (B.4). Since all the states are equally unstable, we let $k_{rj} = k_{r+1j} = \cdots = k_{sj} =: k_{ij}$. Following the same steps as in (B.6), we can show that $x_{s,t}$ is stable if $\log\left(|\tilde{\lambda}|\right) < \sum_{j=1}^m k_{ij}C_j$. For $i = r, \ldots, s-1$, if we assume that $x_{i+1,t}$ is stable, the second moment of $x_{i,t+K}$ at $t = lK$ can be bounded as

$$\begin{aligned} \mathbb{E}\left[x^2_{i,t+K}\right] &\overset{(a)}{=} \tilde{\lambda}^2 \mathbb{E}\left[\left(x_{i,t} - \hat{x}_{i,t}\right)^2\right] + \mathbb{E}\left[\left(x_{i+1,t} - \hat{x}_{i+1,t}\right)^2\right] \\ &\quad + 2\tilde{\lambda}\mathbb{E}\left[\left(x_{i,t} - \hat{x}_{i,t}\right)\left(x_{i+1,t} - \hat{x}_{i+1,t}\right)\right] + \nu_i \\ &\overset{(b)}{=} \tilde{\lambda}^2 2^{-2\sum_{j=1}^m k_{ij}C_j} \mathbb{E}\left[x^2_{i,t}\right] + 2^{-2\sum_{j=1}^m k_{ij}C_j} \mathbb{E}\left[x^2_{i+1,t}\right] \\ &\quad + 2\tilde{\lambda}\mathbb{E}\left[\left(x_{i,t} - \hat{x}_{i,t}\right)\left(x_{i+1,t} - \hat{x}_{i+1,t}\right)\right] + \nu_i \\ &\overset{(c)}{\le} \tilde{\lambda}^2 2^{-2\sum_{j=1}^m k_{ij}C_j} \mathbb{E}\left[x^2_{i,t}\right] + 2^{-2\sum_{j=1}^m k_{ij}C_j} \mathbb{E}\left[x^2_{i+1,t}\right] \\ &\quad + 2\tilde{\lambda}\sqrt{\mathbb{E}\left[\left(x_{i,t} - \hat{x}_{i,t}\right)^2\right]\mathbb{E}\left[\left(x_{i+1,t} - \hat{x}_{i+1,t}\right)^2\right]} + \nu_i \\ &= \tilde{\lambda}^2 2^{-2\sum_{j=1}^m k_{ij}C_j} \mathbb{E}\left[x^2_{i,t}\right] + 2^{-2\sum_{j=1}^m k_{ij}C_j} \mathbb{E}\left[x^2_{i+1,t}\right] \\ &\quad + 2\tilde{\lambda} 2^{-2\sum_{j=1}^m k_{ij}C_j} \sqrt{\mathbb{E}\left[x^2_{i,t}\right]\mathbb{E}\left[x^2_{i+1,t}\right]} + \nu_i \\ &\overset{(d)}{\le} b_1 \mathbb{E}\left[x^2_{i,t}\right] + b_2\sqrt{\mathbb{E}\left[x^2_{i,t}\right]} + b_3, \end{aligned} \tag{B.12}$$

where (a) follows from (B.4); (b) follows from (8); (c) follows by Cauchy–Schwarz inequality; (d) follows from the assumption that $x_{i+1,t}$ is stable i.e., $\mathbb{E}\left[x^2_{i+1,t}\right] < M$ (we have already shown that $x_{s,t}$ is stable if $\log\left(|\tilde{\lambda}|\right) < \sum_{j=1}^m k_{ij}C_j$) and by defining $b_1 := \tilde{\lambda}^2 2^{-2\sum_{j=1}^m k_{ij}C_j}$, $b_2 := 2\tilde{\lambda}2^{-2\sum_{j=1}^m k_{ij}C_j}\sqrt{M}$, and $b_3 := 2^{-2\sum_{j=1}^m k_{ij}C_j}M + \nu_i$. We now find a condition to ensure convergence of the sequence,

$$\alpha_{t+1} = b_1\alpha_t + b_2\sqrt{\alpha_t} + b_3, \tag{B.13}$$

by making use of the following lemma.

Lemma B.1 ([36, Lemma 6.1]). *Let $T : \mathbb{R} \mapsto \mathbb{R}$ be a non-decreasing continuous mapping with a unique fixed point $x^\star \in \mathbb{R}$. If there exists $u \le x^\star \le v$ such that $T(u) \ge u$ and $T(v) \le v$, then the sequence generated by $x_{t+1} = T(x_t)$, $t \in \mathbb{N}$ converges starting from any initial value $x_0 \in \mathbb{R}$.*

We observe that the mapping $T(\alpha) = b_1\alpha + b_2\sqrt{\alpha} + b_3$ with $\alpha \ge 0$ is monotonically increasing since $b_1, b_2 > 0$. It will have a unique fixed point $\alpha^\star$ if and only if $b_1 < 1$, since $b_2, b_3 > 0$. Assuming that $b_1 < 1$, there exists $u < \alpha^\star < v$ such that $T(u) \ge u$ and $T(v) \le v$. Therefore by Lemma B.1, the sequence $\{\alpha_t\}$ is convergent if $b_1 = \tilde{\lambda}^2 2^{-2\sum_{j=1}^m k_{ij}C_j} < 1 \Rightarrow \log(\tilde{\lambda}) < \sum_{j=1}^m k_{ij}C_j$. Since $|\tilde{\lambda}_i| = \tilde{\lambda}$, $x_{i,t}$ is stable if

$$\log\left(|\tilde{\lambda}_i|\right) < \sum_{j=1}^m k_{ij}C_j. \tag{B.14}$$

$$
\begin{aligned}
\mathbb{E}\left[x_{i,t+K}^2\right]+\mathbb{E}\left[x_{i+1,t+K}^2\right] &\overset{(a)}{=} (\tilde{\sigma}^2+\tilde{\omega}^2)\,2^{-2\sum_{j=1}^{m}k_{ij}C_j}\left(\mathbb{E}\left[x_{i,t}^2\right]+\mathbb{E}\left[x_{i+1,t}^2\right]\right)+2^{-2\sum_{j=1}^{m}k_{ij}C_j}\left(\mathbb{E}\left[x_{i+2,t}^2\right]+\mathbb{E}\left[x_{i+3,t}^2\right]\right)\\
&\quad+2\tilde{\sigma}\mathbb{E}\left[\left(x_{i,t}-\hat{x}_{i,t}\right)\left(x_{i+2,t}-\hat{x}_{i+2,t}\right)\right]+2\tilde{\sigma}\mathbb{E}\left[\left(x_{i+1,t}-\hat{x}_{i+1,t}\right)\left(x_{i+3,t}-\hat{x}_{i+3,t}\right)\right]+2\tilde{\omega}\mathbb{E}\left[\left(x_{i+1,t}-\hat{x}_{i+1,t}\right)\left(x_{i+2,t}-\hat{x}_{i+2,t}\right)\right]\\
&\quad-2\tilde{\omega}\mathbb{E}\left[\left(x_{i,t}-\hat{x}_{i,t}\right)\left(x_{i+3,t}-\hat{x}_{i+3,t}\right)\right]+\nu_i+\nu_{i+1}\\
&\overset{(b)}{\leq} (\tilde{\sigma}^2+\tilde{\omega}^2)\,2^{-2\sum_{j=1}^{m}k_{ij}C_j}\left(\mathbb{E}\left[x_{i,t}^2\right]+\mathbb{E}\left[x_{i+1,t}^2\right]\right)+2^{-2\sum_{j=1}^{m}k_{ij}C_j}\left(\mathbb{E}\left[x_{i+2,t}^2\right]+\mathbb{E}\left[x_{i+3,t}^2\right]\right)\\
&\quad+2\tilde{\sigma}2^{-2\sum_{j=1}^{m}k_{ij}C_j}\left(\sqrt{\mathbb{E}\left[x_{i,t}^2\right]\mathbb{E}\left[x_{i+2,t}^2\right]}+\sqrt{\mathbb{E}\left[x_{i+1,t}^2\right]\mathbb{E}\left[x_{i+3,t}^2\right]}\right)\\
&\quad+2\tilde{\omega}2^{-2\sum_{j=1}^{m}k_{ij}C_j}\left(\sqrt{\mathbb{E}\left[x_{i+1,t}^2\right]\mathbb{E}\left[x_{i+2,t}^2\right]}+\sqrt{\mathbb{E}\left[x_{i,t}^2\right]\mathbb{E}\left[x_{i+3,t}^2\right]}\right)+\nu_i+\nu_{i+1}\\
&\overset{(c)}{\leq} (\tilde{\sigma}^2+\tilde{\omega}^2)\,2^{-2\sum_{j=1}^{m}k_{ij}C_j}\left(\mathbb{E}\left[x_{i,t}^2\right]+\mathbb{E}\left[x_{i+1,t}^2\right]\right)+2^{-2\sum_{j=1}^{m}k_{ij}C_j}M+2\tilde{\sigma}2^{-2\sum_{j=1}^{m}k_{ij}C_j}M\left(\sqrt{\mathbb{E}\left[x_{i,t}^2\right]}+\sqrt{\mathbb{E}\left[x_{i+1,t}^2\right]}\right)\\
&\quad+2\tilde{\omega}2^{-2\sum_{j=1}^{m}k_{ij}C_j}M\left(\sqrt{\mathbb{E}\left[x_{i,t}^2\right]}+\sqrt{\mathbb{E}\left[x_{i+1,t}^2\right]}\right)+\nu_i+\nu_{i+1}\\
&\overset{(d)}{\leq} b_1\left(\mathbb{E}\left[x_{i,t}^2\right]+\mathbb{E}\left[x_{i+1,t}^2\right]\right)+b_2\sqrt{\mathbb{E}\left[x_{i,t}^2\right]+\mathbb{E}\left[x_{i+1,t}^2\right]}+b_3 \qquad (\star)
\end{aligned}
$$

Box I.

Finally, let us consider the states given in (B.4) corresponding to the Jordan block associated with the complex eigenvalues. Since all the state components $\{x_{i,t}\}_{i=r}^{s-1}$ are equally unstable, we fix $k_{rj} = k_{r+1j} = \cdots = k_{sj} = k_{ij}$. Following the same steps as in (B.8), (B.9), and (B.10), we can show that the sum $\mathbb{E}\left[x_{i,t+K}^2\right]+\mathbb{E}\left[x_{i+1,t+K}^2\right]$ is bounded if

$$\log\left(|\tilde{\lambda}_{s-1}|\right) < \sum_{j=1}^{m} k_{ij}C_j, \qquad \log\left(|\tilde{\lambda}_{s}|\right) < \sum_{j=1}^{m} k_{ij}C_j. \tag{B.15}$$

For $i = r, r+2, r+4, \ldots, s-3$, we have

$$
\begin{aligned}
\mathbb{E}\left[x_{i,t+K}^2\right] &= 2^{-2\sum_{j=1}^{m}k_{ij}C_j}\left(\tilde{\sigma}^2\mathbb{E}\left[x_{i,t}^2\right]+\tilde{\omega}^2\mathbb{E}\left[x_{i+1,t}^2\right]+\mathbb{E}\left[x_{i+2,t}^2\right]\right)\\
&\quad+\,2\tilde{\sigma}\tilde{\omega}\mathbb{E}\left[\left(x_{i,t}-\hat{x}_{i,t}\right)\left(x_{i+1,t}-\hat{x}_{i+1,t}\right)\right]\\
&\quad+\,2\tilde{\omega}\mathbb{E}\left[\left(x_{i+1,t}-\hat{x}_{i+1,t}\right)\left(x_{i+2,t}-\hat{x}_{i+2,t}\right)\right]\\
&\quad+\,2\tilde{\sigma}\mathbb{E}\left[\left(x_{i,t}-\hat{x}_{i,t}\right)\left(x_{i+2,t}-\hat{x}_{i+2,t}\right)\right]+\nu_i,
\end{aligned}\tag{B.16}
$$

$$
\begin{aligned}
\mathbb{E}\left[x_{i+1,t+K}^2\right] &= 2^{-2\sum_{j=1}^{m}k_{ij}C_j}\left(\tilde{\omega}^2\mathbb{E}\left[x_{i,t}^2\right]+\tilde{\sigma}^2\mathbb{E}\left[x_{i+1,t}^2\right]+\mathbb{E}\left[x_{i+3,t}^2\right]\right)\\
&\quad-\,2\tilde{\sigma}\tilde{\omega}\mathbb{E}\left[\left(x_{i,t}-\hat{x}_{i,t}\right)\left(x_{i+1,t}-\hat{x}_{i+1,t}\right)\right]\\
&\quad+\,2\tilde{\sigma}\mathbb{E}\left[\left(x_{i+1,t}-\hat{x}_{i+1,t}\right)\left(x_{i+3,t}-\hat{x}_{i+3,t}\right)\right]\\
&\quad-\,2\tilde{\omega}\mathbb{E}\left[\left(x_{i,t}-\hat{x}_{i,t}\right)\left(x_{i+3,t}-\hat{x}_{i+3,t}\right)\right]+\nu_{i+1}.
\end{aligned}\tag{B.17}
$$

Assuming that $x_{i+2,t}$ and $x_{i+3,t}$ are stable, the sum $\mathbb{E}\left[x_{i,t+K}^2\right]+\mathbb{E}\left[x_{i+1,t+K}^2\right]$ is bounded by ($\star$) given in Box I, where (a) follows from (B.16) and (B.17); (b) follows from the Cauchy–Schwarz inequality; (c) follows from the assumption that $\left(\mathbb{E}\left[x_{i+2,t}^2\right]+\mathbb{E}\left[x_{i+3,t}^2\right]\right) < M$, and ($d$) follows by using the following inequality $\sqrt{\mathbb{E}\left[x_{i,t}^2\right]}+\sqrt{\mathbb{E}\left[x_{i+1,t}^2\right]} < 2\sqrt{\mathbb{E}\left[x_{i,t}^2\right]+\mathbb{E}\left[x_{i+1,t}^2\right]}$ and defining $b_1 := (\tilde{\sigma}^2+\tilde{\omega}^2)\, 2^{-2\sum_{j=1}^{m}k_{ij}C_j}$, $b_2 := 8\,(\tilde{\sigma}+\tilde{\omega})\,2^{-2\sum_{j=1}^{m}k_{ij}C_j}M$, and $b_3 := 2^{-2\sum_{j=1}^{m}k_{ij}C_j}M+\nu_i+\nu_{i+1}$. If we define $\alpha_t := \mathbb{E}\left[x_{i,t}^2\right]+\mathbb{E}\left[x_{i+1,t}^2\right]$, then according to ($\star$) we get a majorizing sequence that has the same form as (B.13) with the values of b_i given above. Using Lemma B.1 we can show that α_t is convergent if $b_1 = (\tilde{\sigma}^2+\tilde{\omega}^2)\,2^{-2\sum_{j=1}^{m}k_{ij}C_j} < 1$. Since $(\tilde{\sigma}^2+\tilde{\omega}^2) = |\tilde{\lambda}|^2 = |\tilde{\lambda}_i|^2$, we get

$$\log\left(|\tilde{\lambda}_i|\right) < \sum_{j=1}^{m} k_{ij}C_j. \tag{B.18}$$

According to (B.7), (B.11), (B.14), and (B.18), all modes are stable if

$$\log\left(|\tilde{\lambda}_i|\right) < \sum_{j=1}^{m} k_{ij}C_j. \tag{B.19}$$

Since $|\tilde{\lambda}_i| = |\lambda_i|^K$, we can re-write (B.19) as,

$$\log\left(|\lambda_i|^K\right) < \sum_{j=1}^{m} k_{ij}C_j \Rightarrow \log\left(|\lambda_i|\right) < \sum_{j=1}^{m} f_{ij}C_j, \tag{B.20}$$

where $f_{ij} := \frac{k_{ij}}{K}$. According to (7) $f_{ij} \geq 0$, $\sum_{j=1}^{m} f_{ij} \leq 1$, and $\sum_{i=1}^{n} f_{ij} \leq 1$. This completes the proof.

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