4
Linear Systems

The problem of finding the solutions to a system of linear equations provides the theoretical and computational foundation for linear algebra. In this chapter, we introduce linear systems and elementary row operations, we repackage the features of a linear system in a matrix, and we present an algorithm for solving linear systems.

4.0 Systems of Linear Equations

HOW CAN WE FIND THE INTERSECTION OF A COLLECTION OF HYPERPLANES? A ***linear equation*** in the variables $x_1, x_2, \ldots, x_n$ is an equation of the form $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$ where ***constant term*** is $b \in \mathbb{K}$ and the ***coefficients*** are the scalars $a_1, a_2, \ldots, a_n \in \mathbb{K}$. A ***linear system***, short for a ***system of linear equations***, is a finite collection of linear equations. A ***solution*** to a linear system is a vector $\vec{v} := v_1\,\vec{e}_1 + v_2\,\vec{e}_2 + \cdots + v_n\,\vec{e}_n \in \mathbb{K}^n$ such that the corresponding point, relative to the origin, lies on each hyperplane in the collection. The ***solution set*** of a linear system is the set of all solutions. The phrase "to solve the system" means to explicitly describe the solution set.

4.0.0 Problem. Solve the linear system $\left\{ \begin{array}{r} x - 2y = -1 \\ -x + 3y = 3 \end{array} \right\}$.

Geometric solution. These lines intersect only at the point $(3, 2)$. □

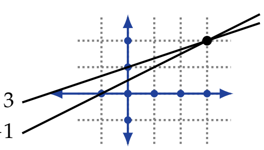

Figure 4.0: Intersection of two lines

4.0.1 Problem. Solve the linear system $\left\{ \begin{array}{r} x - 2y = -1 \\ -x + 2y = 3 \end{array} \right\}$.

Geometric solution. The lines are parallel, so there is no solution. □

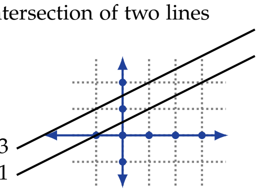

Figure 4.1: Two parallel lines

4.0.2 Problem. Solve the linear system $\left\{ \begin{array}{r} x - 2y = -1 \\ -x + 2y = 1 \end{array} \right\}$.

Geometric solution. The lines coincide, so every point on one line belongs to the solution set. Since the points $(-1, 0)$ and $(1, 1)$ lie on both lines, the solution set can be described as

$$\big\{ (1-t)(-\vec{e}_1) + t(\vec{e}_1 + \vec{e}_2) \mid t \in \mathbb{K} \big\} = \big\{ (2t-1, t) \mid t \in \mathbb{K} \big\} .$$

Alternatively, we also have

$$\big\{ (1-s)\big(\tfrac{1}{2}\,\vec{e}_2\big) + s(\vec{e}_1 + \vec{e}_2) \mid s \in \mathbb{K} \big\} = \big\{ \big(s, \tfrac{1}{2}(s+1)\big) \mid s \in \mathbb{K} \big\} ,$$

because the points $(0, 1/2)$ and $(1, 1)$ also lie on both lines. □

4.0.3 Definition. A linear system is ***consistent*** when it has at least one solution; it is ***inconsistent*** when it has no solutions. Geometrically, a linear system is consistent if its hyperplanes have a nonempty intersection.

HOW CAN WE FIND THE SOLUTION SET OF A LINEAR SYSTEM? Two linear systems are ***equivalent*** if they have the same solution set. The basic strategy to solve a linear system is to recursively replace a given linear system with another equivalent one that is easier to solve. The following three operations, called ***elementary row operations***, produce equivalent linear systems. As an abbreviation, we use the vector $\vec{r}_j$ to denote the j-th equation in a linear system.

A linear system is most often given by writing each linear equation on a separate row.

(row add) Replace one equation by the sum of that equation and a scalar multiple of another. Symbolically, we have $\vec{r}_j + c\,\vec{r}_k \mapsto \vec{r}_j$ for some $j \neq k$ and $c \in \mathbb{K}$.

(row swap) Interchange two equations or $\left\{ \begin{smallmatrix} \vec{r}_j \mapsto \vec{r}_k \\ \vec{r}_k \mapsto \vec{r}_j \end{smallmatrix} \right\}$ for some $j \neq k$.

(row multiple) Multiply all the terms in an equation by a nonzero constant or $c\,\vec{r}_j \mapsto \vec{r}_j$ for some $0 \neq c \in \mathbb{K}$.

As we will see, the elementary row operations produce equivalent linear systems because they are "invertible"; they can be undone by another elementary operation.

4.0.4 Problem. Solve the linear system $\left\{ \begin{aligned} x - 2y + z &= 0 \\ 2y - 8z &= 8 \\ -4x + 5y + 9z &= -9 \end{aligned} \right\}$.

Solution. Elementary row operations give

$$\left\{ \begin{aligned} x - 2y + z &= 0 \\ 2y - 8z &= 8 \\ -4x + 5y + 9z &= -9 \end{aligned} \right\} \xrightarrow{\frac{1}{2}\vec{r}_2 \mapsto \vec{r}_2} \left\{ \begin{aligned} x - 2y + z &= 0 \\ y - 4z &= 4 \\ -4x + 5y + 9z &= -9 \end{aligned} \right\} \xrightarrow{\vec{r}_3 + 4\vec{r}_1 \mapsto \vec{r}_3} \left\{ \begin{aligned} x - 2y + z &= 0 \\ y - 4z &= 4 \\ -3y + 13z &= -9 \end{aligned} \right\}$$

$$\xrightarrow{\vec{r}_3 + 3\vec{r}_2 \mapsto \vec{r}_3} \left\{ \begin{aligned} x - 2y + z &= 0 \\ y - 4z &= 4 \\ z &= 3 \end{aligned} \right\} \xrightarrow{\vec{r}_2 + 4\vec{r}_3 \mapsto \vec{r}_2} \left\{ \begin{aligned} x - 2y + z &= 0 \\ y &= 16 \\ z &= 3 \end{aligned} \right\}$$

$$\xrightarrow{\vec{r}_1 - \vec{r}_3 \mapsto \vec{r}_1} \left\{ \begin{aligned} x - 2y &= -3 \\ y &= 16 \\ z &= 3 \end{aligned} \right\} \xrightarrow{\vec{r}_1 + 2\vec{r}_2 \mapsto \vec{r}_1} \left\{ \begin{aligned} x &= 29 \\ y &= 16 \\ z &= 3 \end{aligned} \right\}.$$

Thus, the unique solution is the point $(29, 16, 3) \in \mathbb{R}^3$. □

Verification. We have $\left\{ \begin{aligned} (29) - 2(16) + (3) &= 0 \\ 2(16) - 8(3) &= 8 \\ -4(29) + 5(16) + 9(3) &= -9 \end{aligned} \right\}$. □

4.0.5 Problem. Solve the linear system $\left\{ \begin{aligned} y - 4z &= 8 \\ 2x - 3y + 2z &= 1 \\ 5x - 8y + 7z &= 1 \end{aligned} \right\}$.

Solution. Elementary row operations give

$$\left\{ \begin{aligned} y - 4z &= 8 \\ 2x - 3y + 2z &= 1 \\ 5x - 8y + 7z &= 1 \end{aligned} \right\} \xrightarrow{\vec{r}_3 - \frac{5}{2}\vec{r}_2 \mapsto \vec{r}_3} \left\{ \begin{aligned} y - 4z &= 8 \\ 2x - 3y + 2z &= 1 \\ -\tfrac{1}{2}y + 2z &= -\tfrac{3}{2} \end{aligned} \right\} \xrightarrow{\vec{r}_3 + \frac{1}{2}\vec{r}_1 \mapsto \vec{r}_3} \left\{ \begin{aligned} y - 4z &= 8 \\ 2x - 3y + 2z &= 1 \\ 0 &= \tfrac{5}{2} \end{aligned} \right\},$$

so these equivalent systems have no solutions and the original system is inconsistent. □

4.0.6 Problem. Solve the linear system $\left\{\begin{array}{rcrcrl} \sqrt{2}\,\mathrm{i}\,x & + & y & & & = 0 \\ -x & + & \sqrt{2}\,\mathrm{i}\,y & + & z & = 0 \\ & & -y & + & \sqrt{2}\,\mathrm{i}\,z & = 0 \end{array}\right\}$.

Solution. Elementary row operations give

$$\left\{\begin{array}{rcrcrl} \sqrt{2}\,\mathrm{i}\,x & + & y & & & = 0 \\ -x & + & \sqrt{2}\,\mathrm{i}\,y & + & z & = 0 \\ & & -y & + & \sqrt{2}\,\mathrm{i}\,z & = 0 \end{array}\right\} \xrightarrow{\vec{r}_1 + \sqrt{2}\,\mathrm{i}\,\vec{r}_2 \mapsto \vec{r}_1} \left\{\begin{array}{rcrcrl} & & -y & + & \sqrt{2}\,\mathrm{i}\,z & = 0 \\ -x & + & \sqrt{2}\,\mathrm{i}\,y & + & z & = 0 \\ & & -y & + & \sqrt{2}\,\mathrm{i}\,z & = 0 \end{array}\right\}$$

$$\xrightarrow{\vec{r}_3 - \vec{r}_1 \mapsto \vec{r}_3} \left\{\begin{array}{rcrcrl} & & -y & + & \sqrt{2}\,\mathrm{i}\,z & = 0 \\ -x & + & \sqrt{2}\,\mathrm{i}\,y & + & z & = 0 \\ & & & & 0 & = 0 \end{array}\right\} \xrightarrow{\vec{r}_2 + \sqrt{2}\,\mathrm{i}\,\vec{r}_1 \mapsto \vec{r}_2} \left\{\begin{array}{rcrcrl} & & -y & + & \sqrt{2}\,\mathrm{i}\,z & = 0 \\ -x & & & - & z & = 0 \\ & & & & 0 & = 0 \end{array}\right\}$$

$$\xrightarrow{-\vec{r}_1 \mapsto \vec{r}_1} \left\{\begin{array}{rcrcrl} & & y & - & \sqrt{2}\,\mathrm{i}\,z & = 0 \\ -x & & & - & z & = 0 \\ & & & & 0 & = 0 \end{array}\right\} \xrightarrow{-\vec{r}_2 \mapsto \vec{r}_2} \left\{\begin{array}{rcrcrl} & & y & - & \sqrt{2}\,\mathrm{i}\,z & = 0 \\ x & & & + & z & = 0 \\ & & & & 0 & = 0 \end{array}\right\} \longrightarrow \left\{\begin{array}{rr} y = & \sqrt{2}\,\mathrm{i}\,z \\ x = & -z \\ 0 = & 0 \end{array}\right\},$$

so the solutions set is

$$\left\{ -t\,\vec{e}_1 + \sqrt{2}\,\mathrm{i}\,t\,\vec{e}_2 + t\,\vec{e}_3 \mid t \in \mathbb{C} \right\} = \left\{ t \begin{bmatrix} -1 \\ \sqrt{2}\mathrm{i} \\ 1 \end{bmatrix} \;\middle|\; t \in \mathbb{C} \right\}. \qquad \square$$

Verification. We have $\left\{\begin{array}{rcrcrcl} \sqrt{2}\,\mathrm{i}(-t) & + & (\sqrt{2}\,\mathrm{i}\,t) & & & = & 0 \\ -(-t) & + & \sqrt{2}\,\mathrm{i}\,(\sqrt{2}\,\mathrm{i}\,t) & + & (t) & = & 0 \\ & & -(\sqrt{2}\,\mathrm{i}\,t) & + & \sqrt{2}\,\mathrm{i}(t) & = & 0 \end{array}\right\}$. □

Exercises

4.0.7 Problem. Determine which of the following statements are true. If a statement is false, then provide a counterexample.

i. In a linear system, the same equation can appear more than once.
ii. A linear system must have more than one linear equation.
iii. In a linear system, the number of equations must equal the number of variables.
iv. All linear systems have a nonempty solution set.
v. A consistent linear system must have exactly one solution.
vi. A linear system is inconsistent if it has infinitely many solutions.
vii. There are four types of elementary row operations.
viii. Multiplying an equation by 0 produces an equivalent linear system.

4.0.8 Problem. Solve the linear system $\left\{\begin{array}{rcrcrcr} & & y & + & 4z & = & -5 \\ x & + & 3y & + & 5z & = & -2 \\ 3x & + & 7y & + & 7z & = & 6 \end{array}\right\}$.

4.0.9 Problem. Solve the linear system

$$\left\{ \begin{array}{rcrcrcl} \mathrm{i}\, z_1 & + & (1+\mathrm{i})\, z_2 & & & = & \mathrm{i} \\ (1-\mathrm{i})\, z_1 & + & z_2 & - & \mathrm{i}\, z_3 & = & 1 \\ & & \mathrm{i}\, z_2 & + & z_3 & = & 1 \end{array} \right\} .$$

4.0.10 Problem. Establish that the elementary row operations are invertible by proving the following identities.

i. For all $j \neq k$ and all $c \in \mathbb{K}$, show that the row add operations $\vec{r}_j + c\,\vec{r}_k \mapsto \vec{r}_j$ an $\vec{r}_j - c\,\vec{r}_k \mapsto \vec{r}_j$ compose to the identity in either order.

ii. For all $j \neq k$, show that the row swap operation is involutive; it is its own inverse.

iii. For all $0 \neq c \in \mathbb{K}$, show that the row multiple operations $c\,\vec{r}_j \mapsto \vec{r}_j$ and $\frac{1}{c}\,\vec{r}_j \mapsto \vec{r}_j$ compose to the identity in either order.

4.1 Matrices

HOW CAN WE WORK EFFICIENTLY WITH LINEAR SYSTEMS? For any two nonnegative integers m and n, an $(m \times n)$***-matrix*** is an array of scalars consisting with m rows and n columns:

The mathematical term "matrix" was coined in 1850 by J.J. Sylvester

$$\mathbf{A} := [a_{j,k}] = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & a_{m,3} & \cdots & a_{m,n} \end{bmatrix} .$$

Two matrices are ***equal*** if they have the same number of rows, the same number of columns, and their corresponding entries are all equal. The two important matrices associated to the linear system

$$\left\{ \begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \cdots & + & a_{1,n}x_n & = & b_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \cdots & + & a_{2,n}x_n & = & b_2 \\ \vdots & & \vdots & & \vdots & & & & \vdots \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + & \cdots & + & a_{m,n}x_n & = & b_m \end{array} \right\}$$

are the ***coefficient matrix*** and the ***augmented matrix***

$$\mathbf{A} = [a_{j,k}] = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & b_1 \\ a_{1,1} & a_{1,2} & \cdots & a_{1,n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} & b_m \end{bmatrix} .$$

The elementary row operation introduced for linear systems naturally extend to matrices.

4.1.0 Problem. Solve the linear system $\left\{ \begin{array}{rcrcrcl} 2\,x & + & 2\,y & & & = & 0 \\ x & - & 2\,y & - & \mathrm{i}\,z & = & 0 \\ \mathrm{i}\,x & + & y & + & z & = & 0 \end{array} \right\}$.

Solution. Elementary row operations applied to the augmented matrix associated to this linear system give

$$\begin{bmatrix} 2 & 2 & 0 & 0 \\ 1 & -2 & -\mathrm{i} & 0 \\ \mathrm{i} & 1 & 1 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}\vec{r}_1 \mapsto \vec{r}_1} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -2 & -\mathrm{i} & 0 \\ \mathrm{i} & 1 & 1 & 0 \end{bmatrix} \xrightarrow{\vec{r}_2 - \vec{r}_1 \mapsto \vec{r}_2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -3 & -\mathrm{i} & 0 \\ \mathrm{i} & 1 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{\vec{r}_3 - \mathrm{i}\,\vec{r}_1 \mapsto \vec{r}_3} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -3 & -\mathrm{i} & 0 \\ 0 & 1-\mathrm{i} & 1 & 0 \end{bmatrix} \xrightarrow{(1+\mathrm{i})\,\vec{r}_3 \mapsto \vec{r}_3} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -3 & -\mathrm{i} & 0 \\ 0 & 2 & 1+\mathrm{i} & 0 \end{bmatrix}$$

$$\xrightarrow{\vec{r}_3 + \vec{r}_2 \mapsto \vec{r}_3} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -3 & -\mathrm{i} & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \xrightarrow{\vec{r}_2 - 3\,\vec{r}_3 \mapsto \vec{r}_2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & -3-\mathrm{i} & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{(-3-\mathrm{i})^{-1}\,\vec{r}_2 \mapsto \vec{r}_2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \xrightarrow{\vec{r}_1 + \vec{r}_3 \mapsto \vec{r}_1} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{\vec{r}_1 - \vec{r}_2 \mapsto \vec{r}_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \xrightarrow{\vec{r}_3 - \vec{r}_2 \mapsto \vec{r}_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-\vec{r}_3 \mapsto \vec{r}_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow[\vec{r}_3 \mapsto \vec{r}_2]{\vec{r}_2 \mapsto \vec{r}_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} .$$

Hence, the given linear system is equivalent to $\left\{ \begin{smallmatrix} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{smallmatrix} \right\}$, so the unique solution is $\vec{\mathbf{0}} = \left[\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix} \right]$. □

The next definition describes the "simplest" matrices obtained via elementary row operations.

4.1.1 Definition. The ***leading entry*** of a row refers to the leftmost nonzero entry. An $(m \times n)$-matrix is in ***reduced row echelon form*** if it satisfies the following properties:

- the first r rows are nonzero and the last $m - r$ rows are zero;
- the leading entry in any row is always to the right of the leading entry of the row above it or, equivalently, when the leading entry in the j-th row lies in the k_j-th column, we have $k_1 < k_2 < \cdots < k_r$ for all $1 \leqslant j \leqslant r$;
- the leading entry in each nonzero row is 1;
- each leading 1 is the only nonzero entry in its column.

The reduced row echelon can be visualized as

$$\begin{bmatrix} 0 & 0 & \cdots & 1 & * & * & \cdots & * & 0 & * & * & \cdots & * & 0 & * & * & \cdots & * \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & * & * & \cdots & * & 0 & * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & * & * & \cdots & * \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} ,$$

where $*$ denotes an arbitrary scalar.

$$\begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}, \quad \begin{bmatrix} 1&*&*&* \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}, \quad \begin{bmatrix} 0&1&*&* \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix},$$

$$\begin{bmatrix} 0&0&1&* \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}, \quad \begin{bmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}, \quad \begin{bmatrix} 1&0&*&* \\ 0&1&*&* \\ 0&0&0&0 \end{bmatrix},$$

$$\begin{bmatrix} 1&*&0&* \\ 0&0&1&* \\ 0&0&0&0 \end{bmatrix}, \quad \begin{bmatrix} 1&*&*&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{bmatrix}, \quad \begin{bmatrix} 0&1&0&* \\ 0&0&1&* \\ 0&0&0&0 \end{bmatrix},$$

$$\begin{bmatrix} 0&1&*&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{bmatrix}, \quad \begin{bmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{bmatrix}, \quad \begin{bmatrix} 1&0&0&* \\ 0&1&0&* \\ 0&0&1&* \end{bmatrix},$$

$$\begin{bmatrix} 1&0&*&0 \\ 0&1&*&0 \\ 0&0&0&1 \end{bmatrix}, \quad \begin{bmatrix} 1&*&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix}, \quad \begin{bmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix}.$$

Table 4.1: The distinct reduced row echelon forms for a (3×4)-matrix

The next subsection shows that each matrix is equivalent to a unique matrix in reduced row echelon form and that the reduced row echelon form can be obtained by a (non-unique) sequence of elementary row operations.

4.1.2 Problem. Find the reduced row echelon form of the matrix

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}.$$

Solution. Elementary row operations give

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ \underline{3} & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \xrightarrow{\frac{1}{3}\vec{r}_3 \mapsto \vec{r}_3} \begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ \underline{1} & -3 & 4 & -3 & 2 & 5 \end{bmatrix} \xrightarrow[\vec{r}_1 \mapsto \vec{r}_3]{\vec{r}_3 \mapsto \vec{r}_1} \begin{bmatrix} \underline{1} & -3 & 4 & -3 & 2 & 5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\xrightarrow{\vec{r}_2 - 3\vec{r}_1 \mapsto \vec{r}_2} \begin{bmatrix} \underline{1} & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \xrightarrow{\frac{1}{2}\vec{r}_2 \mapsto \vec{r}_2} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & \underline{1} & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \xrightarrow{\vec{r}_1 + 3\vec{r}_2 \mapsto \vec{r}_1} \begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & \underline{1} & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\xrightarrow{\vec{r}_3 - 3\vec{r}_2 \mapsto \vec{r}_3} \begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & \underline{1} & 4 \end{bmatrix} \xrightarrow{\vec{r}_1 - 5\vec{r}_3 \mapsto \vec{r}_1} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & \underline{1} & 4 \end{bmatrix} \xrightarrow{\vec{r}_2 - \vec{r}_3 \mapsto \vec{r}_2} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}.$$

□

Exercises

4.1.3 Problem. Determine which of the following statements are true. If a statement is false, then provide a counterexample.

i. A matrix must have the same number of row and columns.
ii. The matrix entry $a_{j,k}$ lies in the jth row and the kth column.
iii. For a given linear system, the associated augmented matrix has one more row than the associated coefficient matrix.
iv. If a linear system has m equations and n variables, then the associated augmented matrix has m rows and $n+1$ columns.
v. The zero matrix is in reduced row echelon form.

4.1.4 Problem. Find the reduced row echelon form of the matrix

$$\begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 5 \\ 1 & -1 & -3 \\ 1 & 0 & -1 \end{bmatrix}.$$

4.1.5 Problem. List all of the distinct reduced row echelon forms for a (4×6)-matrix.

4.1.6 Problem. Examine compositions of the row add operations.

i. For all $j \neq k$ and all $c, d \in \mathbb{K}$, show that the composition, in either order, of row add operations $\vec{r}_j + c\,\vec{r}_k \mapsto \vec{r}_j$ and $\vec{r}_j + d\,\vec{r}_k \mapsto \vec{r}_j$ equals the row add operation $\vec{r}_j + (c+d)\,\vec{r}_k \mapsto \vec{r}_j$.