5.2 Matrix Arithmetic

WHAT OPERATIONS DO MATRICES ACQUIRE FROM THEIR FIELD OF SCALARS? Extending the basic operations on column vectors, addition and scalar multiplication of matrices are also defined entrywise.

5.2.0 Definition. Let m and n be nonnegative integers. For any two $(m \times n)$-matrices $\mathbf{A}$ and $\mathbf{B}$, the ***sum*** $\mathbf{A} + \mathbf{B}$ is the $(m \times n)$-matrix whose entries are the sum of the corresponding entries in $\mathbf{A}$ and $\mathbf{B}$. For all $1 \leqslant j \leqslant m$ and all $1 \leqslant k \leqslant n$, let $a_{j,k} \in \mathbb{K}$ and $b_{j,k} \in \mathbb{K}$ denote the (j,k)-entries in $\mathbf{A}$ and $\mathbf{B}$ respectively. The (j,k)-entry in the matrix sum $\mathbf{A} + \mathbf{B}$ is $a_{j,k} + b_{j,k}$.

For instance, we have

$$\begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 4+1 & 0+1 & 5+1 \\ 3+(-1) & 3+5 & 2+7 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{bmatrix}.$$

5.2.1 Definition. For any scalar $c \in \mathbb{K}$ and any matrix $\mathbf{A}$, the ***scalar multiple*** $c\,\mathbf{A}$ is the matrix whose entries are c times the corresponding entry in $\mathbf{A}$. When $a_{j,k} \in \mathbb{K}$ denotes the (j,k)-entry in $\mathbf{A}$, the (j,k)-entry in $c\,\mathbf{A}$ is $c\,a_{j,k}$.

We illustrate this operation with two examples:

$$2\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix} = \begin{bmatrix} (2)(1) & (2)(1) & (2)(1) \\ (2)(3) & (2)(5) & (2)(7) \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 6 & 10 & 14 \end{bmatrix},$$

$$(-1)\begin{bmatrix} 0 & -6 \\ 6 & -1 \\ 9 & 6 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} (-1)(0) & (-1)(-6) \\ (-1)(6) & (-1)(-1) \\ (-1)(9) & (-1)(6) \\ (-1)(4) & (-1)(1) \end{bmatrix} = \begin{bmatrix} 0 & 6 \\ -6 & 1 \\ -9 & -6 \\ -4 & -1 \end{bmatrix}.$$

Viewing a matrix as a list of columns, two matrices are equal if and only if the corresponding columns are equal, the sum of two matrices equals the sum of the corresponding columns, and the scalar multiple of a matrix is the scalar multiple of its columns.

5.2.2 Problem. Solve the matrix equation $\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} + \mathbf{X} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$.

Solution. We have

$$\mathbf{X} = \left(\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} + \mathbf{X}\right) - \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 0 & 1 \end{bmatrix}. \qquad \square$$

5.2.3 Definition. A matrix whose entries are all zero is called the ***zero matrix***. By an abuse of notation, the zero matrix of any size is denoted by $\mathbf{0}$. One can almost always determine from context the size of the matrix is represented by $\mathbf{0}$.

5.2.4 Problem. Let $\mathbf{A}$ be a matrix and let c be a scalar. Prove that the matrix equation $c\,\mathbf{A} = \mathbf{0}$ implies that $c = 0$ or $\mathbf{A} = \mathbf{0}$.

Solution. For all $1 \leqslant j \leqslant m$ and all $1 \leqslant k \leqslant n$, let $a_{j,k} \in \mathbb{K}$ denote the (j,k)-entry of the matrix $\mathbf{A}$. The matrix equation $c\,\mathbf{A} = \mathbf{0}$ is equivalent to the equations $c\,a_{j,k} = 0$ for all $1 \leqslant j \leqslant m$ and all $1 \leqslant k \leqslant n$. All of these equations hold when $c = 0$. If $c \neq 0$, then it follows that $a_{j,k} = c^{-1}0 = 0$, so we deduce that $\mathbf{A} = \mathbf{0}$. $\square$

5.2.5 Proposition (Properties of matrix arithmetic). *For any three matrices* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ *having the same size and all scalars* $c, d \in \mathbb{K}$, *we have:*

(commutativity)	$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}\,,$
(associativity)	$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})\,,$
(existence of an additive identity)	$\mathbf{A} + \mathbf{0} = \mathbf{A}\,,$
(existence of additive inverses)	$\mathbf{A} + (-\mathbf{A}) = \mathbf{0}\,,$
(compatibility with scalar multiplication)	$c(\mathbf{A} + \mathbf{B}) = (c\,\mathbf{A}) + (c\,\mathbf{B})\,,$
	$(c + d)\mathbf{A} = (c\,\mathbf{A}) + (c\,\mathbf{A})\,.$

Proof. These properties follow from analogous properties for scalars or column vectors; see Subsection 1.2. □

Not all operations on matrices are inherited from operations on scalars. Switching row and column indices gives a new operation.

5.2.6 Definition. Fix positive integers m and n. For any $(m \times n)$-matrix $\mathbf{A}$, the ***transpose*** is the $(n \times m)$-matrix $\mathbf{A}^\mathsf{T}$ obtained by interchanging the rows and columns of $\mathbf{A}$. When $a_{j,k} \in \mathbb{K}$ denotes the (j,k)-entry in $\mathbf{A}$, the scalar $a_{j,k}$ is the (k,j)-entry in $\mathbf{A}^\mathsf{T}$.

One reflects the entries of $\mathbf{A}$ over its main diagonal (which runs from top-left to bottom-right) to obtain $\mathbf{A}^\mathsf{T}$.

For example, when $\mathbf{A} := \begin{bmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{bmatrix}$, we have $\mathbf{A}^\mathsf{T} = \begin{bmatrix} -5 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix}$.

5.2.7 Proposition (Properties of the transpose). *For any two matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *having the same size and all scalars* $c, d \in \mathbb{K}$, *we have:*

(linearity)	$(c\,\mathbf{A} + d\,\mathbf{B})^\mathsf{T} = c(\mathbf{A}^\mathsf{T}) + d(\mathbf{B}^\mathsf{T})\,,$
(involution)	$(\mathbf{A}^\mathsf{T})^\mathsf{T} = \mathbf{A}\,.$

Proof. For all $1 \leqslant j \leqslant m$ and all $1 \leqslant k \leqslant n$, let $a_{j,k} \in \mathbb{K}$ and $b_{j,k} \in \mathbb{K}$ denote the (j,k)-entries of $\mathbf{A}$ and $\mathbf{B}$ respectively. The definition of the transpose and the entrywise arithmetic for matrices give

$$\begin{aligned} (c\,\mathbf{A} + d\,\mathbf{B})^\mathsf{T} &= \big(c\,[a_{j,k}] + d\,[b_{j,k}]\big)^\mathsf{T} = [c\,a_{j,k} + d\,b_{j,k}]^\mathsf{T} \\ &= [c\,a_{k,j} + d\,b_{k,j}] = c\,[a_{k,j}] + d\,[b_{k,j}] = c(\mathbf{A}^\mathsf{T}) + d(\mathbf{B}^\mathsf{T}) \\ (\mathbf{A}^\mathsf{T})^\mathsf{T} &= \big([a_{j,k}]^\mathsf{T}\big)^\mathsf{T} = \big([a_{k,j}]\big)^\mathsf{T} = [a_{j,k}] = \mathbf{A}\,. \end{aligned}$$ □

5.2.8 Remark. By identifying the vectors $\vec{v} \in \mathbb{K}^n$ and $\vec{w} \in \mathbb{K}^n$ with $(n \times 1)$-matrices and scalars with (1×1)-matrices, the dot product can be reinterpreted as matrix multiplication:

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \mathbf{v}^\mathsf{T}\,\mathbf{w}\,.$$

Exercises

5.2.9 Problem. Determine which of the following statements are true. If a statement is false, then provide a counterexample.

i. The sum is defined for any two matrices.
ii. A matrix and any of its scalar multiplies have the same size.
iii. The zero matrix must have the same number of rows and columns.
iv. The phrase "a linear combination of matrices" does not make sense.
v. The transpose of matrix flips it over its main diagonal.
vi. On (1×1)-matrices, the transpose equals the identity operator.

5.2.10 Problem. A matrix that has the same number of rows as columns is called a ***square matrix***. A square matrix $\mathbf{A}$ is ***symmetric*** if $\mathbf{A} = \mathbf{A}^{\mathsf{T}}$. If $\mathbf{A}$ is a square matrix, then show that the matrix $\mathbf{A} + \mathbf{A}^{\mathsf{T}}$ is symmetric.

5.2.11 Problem. Given the matrix equation $\mathbf{A} = 2\mathbf{A}^{\mathsf{T}}$, prove that $\mathbf{A} = \mathbf{0}$.

5.2.12 Problem. A square matrix $\mathbf{A}$ is ***skew-symmetric*** if $\mathbf{A}^{\mathsf{T}} = -\mathbf{A}$.

i. Prove that any square matrix is the sum of a symmetric matrix and a skew-symmetric matrix.
ii. Illustrate part *i* for the the matrix $\begin{bmatrix} 1 & 5 & 8 & 10 \\ 11 & 2 & 6 & 9 \\ 14 & 12 & 3 & 7 \\ 16 & 15 & 13 & 4 \end{bmatrix}$.

6

Solution Spaces

The solution sets to a system of linear equations are essential objects in linear algebra. We classify the solutions of a homogeneous linear system in terms of the rank of the coefficient matrix, describe the general solutions sets for non-homogeneous linear systems, and identify the linear systems with a unique solution.

6.0 Homogeneous Systems

WHICH SPECIAL LINEAR SYSTEMS GUIDE OUR UNDERSTANDING ALL LINEAR SYSTEMS? Linear systems in which all of the constant terms are zero play a distinguished role.

6.0.0 Definition. A linear system is ***homogeneous*** if it has the form $\mathbf{A}\,\vec{x} = \vec{\mathbf{0}}$ for some matrix $\mathbf{A}$ and some column vector of unknowns $\vec{x}$.

6.0.1 Remark. Every homogeneous linear system is consistent, because the zero vector is a solution. Since the solution set to a homogeneous linear system is just the kernel of its coefficient matrix, Lemma 5.1.1 shows that any linear combination of solutions to a homogeneous linear system is also a solution.

6.0.2 Problem. Solve the homogeneous linear system

$$\left\{ \begin{aligned} 2x + 5y - 4z &= 0 \\ -3x - 2y + 4z &= 0 \\ 6x + y - 8z &= 0 \end{aligned} \right\} .$$

Solution. The row reduction algorithm [4.2.0] gives

$$\begin{bmatrix} \underline{3} & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \xrightarrow[\sim]{\substack{\vec{r}_2 + \vec{r}_1 \mapsto \vec{r}_2 \\ \vec{r}_3 - 2\vec{r}_1 \mapsto \vec{r}_3}} \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & \underline{3} & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \xrightarrow[\sim]{\substack{\vec{r}_1 + \frac{5}{3}\vec{r}_2 \mapsto \vec{r}_1 \\ \vec{r}_3 + 3\vec{r}_2 \mapsto \vec{r}_3}} \begin{bmatrix} 3 & 0 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\sim]{\substack{3^{-1}\vec{r}_1 \mapsto \vec{r}_1 \\ 3^{-1}\vec{r}_2 \mapsto \vec{r}_2}} \begin{bmatrix} \underline{1} & 0 & -\frac{4}{3} & 0 \\ 0 & \underline{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

Since z is the only free variable, the solution set is spanned by the unique fundamental circuit. Hence, the solution set is

$$\operatorname{Ker}\left(\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \right) = \left\{ z \begin{bmatrix} 4/3 \\ 0 \\ \underline{1} \end{bmatrix} \;\middle|\; z \in \mathbb{K} \right\} = \operatorname{Span}\left(\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} \right) ,$$

which is a line passing through the origin. □

6.0.3 Problem. Describe all solutions to $\{10x + 3y - 2z = 0\}$.

Solution. The reduced row echelon form of the coefficient matrix is $[1 \;\; 0.3 \;\; -0.2]$. Hence, there are two fundamental circuits corresponding to the free variables y and z. Hence, the solution set is

$$\left\{ \begin{bmatrix} -0.3y + 0.2z \\ y \\ z \end{bmatrix} \;\middle|\; y, z \in \mathbb{K} \right\} = \left\{ y \begin{bmatrix} -0.3 \\ \underline{1} \\ 0 \end{bmatrix} + z \begin{bmatrix} 0.2 \\ 0 \\ \underline{1} \end{bmatrix} \;\middle|\; y, z \in \mathbb{K} \right\} = \operatorname{Span}\left(\begin{bmatrix} -0.3 \\ \underline{1} \\ 0 \end{bmatrix}, \begin{bmatrix} 0.2 \\ 0 \\ \underline{1} \end{bmatrix} \right),$$

which is a plane passing through the origin. □

For a homogeneous system, the basic numerical invariants of the coefficient matrix nearly determine the solution set. Remember that the rank of a matrix is the number of nonzero rows in its reduced row echelon form.

6.0.4 Proposition. *Consider the homogeneous linear system $\mathbf{A}\,\vec{x} = \vec{\mathbf{0}}$ where the coefficient matrix $\mathbf{A}$ is an $(m \times n)$-matrix having rank r*

i. *When $r = n$, the only solution is the zero vector $\vec{\mathbf{0}} \in \mathbb{K}^n$.*
ii. *When $r < n$, there are infinitely many solutions.*
iii. *When $n > m$, there are infinitely many solutions*

Proof.

i. When $n = r$, every variable is a leading variable and there are no free variables. It follows that, in the reduced row echelon form of the matrix $\mathbf{A}$, the j-th leading one appears in the j-th column. Hence, we have $x_j = 0$ for all $1 \leqslant j \leqslant n$ which means that $\vec{\mathbf{0}}$ is the only solution.

ii. When $n > r$, there are $n - r$ free variables, each of which may be given infinitely many values. The solution space is spanned by the $n - r > 0$ fundamental circuits.

iii. Suppose $m < n$. Since the basic bounds on rank [4.2.3] establish that $r \leqslant m < n$, the number $n - r$ of free variables must be positive. These variables can be given arbitrary values, so there are infinitely many solutions. □

We classify homogeneous systems with a unique solution.

6.0.5 Corollary. *Let $\mathbf{A}$ be an $(m \times n)$-matrix. The homogeneous linear system $\mathbf{A}\,\vec{x} = \vec{\mathbf{0}}$ has the unique solution $\vec{\mathbf{0}}$ if and only if* $\operatorname{rank}(\mathbf{A}) = n$.

Proof. Set $r := \operatorname{rank}(\mathbf{A})$.

$\Rightarrow$: We prove the contrapositive. The basic bounds on rank [4.2.3] gives $r \leqslant n$. If $r \neq n$, then we have $r < n$ and part ii of the proposition shows that the linear system $\mathbf{A}\,\vec{x} = \vec{\mathbf{0}}$ has infinitely many solutions.

$\Leftarrow$: If $r = n$, then part i of the proposition shows the only solution of the linear system $\mathbf{A}\,\vec{x} = \vec{\mathbf{0}}$ is $\vec{x} = \vec{\mathbf{0}}$. □

A conditional statement is logically equivalent to its contrapositive. The ***contrapositive*** of a statement has its hypothesis and its conclusion inverted and flipped: the contrapositive of "P implies Q" is "not Q implies not P".

6.0.6 Problem. Solve the linear system $\mathbf{B}\,\vec{x} = \vec{\mathbf{0}}$ where

$$\mathbf{B} := \begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 2 \\ 1 & -1 & -3 \\ 1 & 0 & -1 \end{bmatrix} .$$

Solution. The row reduction algorithm [4.2.0] gives

$$\begin{bmatrix} \underline{1} & 1 & 2 \\ -1 & 2 & 2 \\ 1 & -1 & -3 \\ 1 & 0 & -1 \end{bmatrix} \xrightarrow[\sim]{\substack{\vec{r}_2 + \vec{r}_1 \mapsto \vec{r}_2 \\ \vec{r}_3 - \vec{r}_1 \mapsto \vec{r}_3 \\ \vec{r}_4 - \vec{r}_1 \mapsto \vec{r}_4}} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 4 \\ 0 & -2 & -5 \\ 0 & \underline{-1} & -3 \end{bmatrix} \xrightarrow[\sim]{\substack{\vec{r}_4 \mapsto \vec{r}_2 \\ \vec{r}_2 \mapsto \vec{r}_4}} \begin{bmatrix} 1 & 1 & 2 \\ 0 & \underline{-1} & -3 \\ 0 & -2 & -5 \\ 0 & 3 & 4 \end{bmatrix} \xrightarrow[\sim]{\substack{\vec{r}_1 + \vec{r}_2 \mapsto \vec{r}_1 \\ \vec{r}_3 - 2\vec{r}_2 \mapsto \vec{r}_3 \\ \vec{r}_4 + 3\vec{r}_2 \mapsto \vec{r}_4}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & -3 \\ 0 & 0 & \underline{1} \\ 0 & 0 & -5 \end{bmatrix}$$

$$\xrightarrow[\sim]{-\vec{r}_2 \mapsto \vec{r}_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & \underline{1} \\ 0 & 0 & -5 \end{bmatrix} \xrightarrow[\sim]{\substack{\vec{r}_1 + \vec{r}_3 \mapsto \vec{r}_1 \\ \vec{r}_2 - 3\vec{r}_3 \mapsto \vec{r}_2 \\ \vec{r}_4 + 5\vec{r}_2 \mapsto \vec{r}_4}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} ,$$

so the only solution is $\vec{\mathbf{0}}$. □

6.0.7 Problem. Solve the linear system $\mathbf{A}\,\vec{x} = \vec{\mathbf{0}}$ where

$$\mathbf{A} := \begin{bmatrix} 1 & 2 & 2 & 1 & 1 & 0 & 5 \\ 1 & 2 & 0 & -1 & -1 & 0 & 1 \\ 1 & 2 & 1 & 0 & 1 & 1 & 2 \\ 2 & 4 & 1 & -1 & 2 & 3 & 1 \\ 2 & 4 & 2 & 0 & 3 & 3 & 3 \end{bmatrix} .$$

Solution. The row reduction algorithm [4.2.0] yields

$$\begin{bmatrix} \underline{1} & 2 & 2 & 1 & 1 & 0 & 5 \\ 1 & 2 & 0 & -1 & -1 & 0 & 1 \\ 1 & 2 & 1 & 0 & 1 & 1 & 2 \\ 2 & 4 & 1 & -1 & 2 & 3 & 1 \\ 2 & 4 & 2 & 0 & 3 & 3 & 3 \end{bmatrix} \xrightarrow[\sim]{\substack{\vec{r}_2 - \vec{r}_1 \mapsto \vec{r}_2 \\ \vec{r}_3 - \vec{r}_1 \mapsto \vec{r}_3 \\ \vec{r}_4 - 2\vec{r}_1 \mapsto \vec{r}_4 \\ \vec{r}_5 - 2\vec{r}_1 \mapsto \vec{r}_5}} \begin{bmatrix} 1 & 2 & 2 & 1 & 1 & 0 & 5 \\ 0 & 0 & \underline{-2} & -2 & -2 & 0 & -4 \\ 0 & 0 & -1 & -1 & 0 & 1 & -3 \\ 0 & 0 & -3 & -3 & 0 & 3 & -9 \\ 0 & 0 & -2 & -2 & 1 & 3 & -7 \end{bmatrix} \xrightarrow[\sim]{-0.5\,\vec{r}_2 \mapsto \vec{r}_2} \begin{bmatrix} 1 & 2 & 2 & 1 & 1 & 0 & 5 \\ 0 & 0 & \underline{1} & 1 & 1 & 0 & 2 \\ 0 & 0 & -1 & -1 & 0 & 1 & -3 \\ 0 & 0 & -3 & -3 & 0 & 3 & -9 \\ 0 & 0 & -2 & -2 & 1 & 3 & -7 \end{bmatrix}$$

$$\xrightarrow[\sim]{\substack{\vec{r}_1 - 2\vec{r}_2 \mapsto \vec{r}_1 \\ \vec{r}_3 + \vec{r}_2 \mapsto \vec{r}_3 \\ \vec{r}_4 + 3\vec{r}_2 \mapsto \vec{r}_4 \\ \vec{r}_5 + 2\vec{r}_2 \mapsto \vec{r}_5}} \begin{bmatrix} 1 & 2 & 0 & -1 & -1 & 0 & 3 \\ 0 & 0 & 1 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & \underline{1} & 1 & -1 \\ 0 & 0 & 0 & 0 & 3 & 3 & -3 \\ 0 & 0 & 0 & 0 & 3 & 3 & -3 \end{bmatrix} \xrightarrow[\sim]{\substack{\vec{r}_1 + \vec{r}_3 \mapsto \vec{r}_1 \\ \vec{r}_2 - \vec{r}_3 \mapsto \vec{r}_2 \\ \vec{r}_4 - 3\vec{r}_3 \mapsto \vec{r}_4 \\ \vec{r}_5 - 3\vec{r}_3 \mapsto \vec{r}_4}} \begin{bmatrix} \underline{1} & 2 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & \underline{1} & 1 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & \underline{1} & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

Since the columns containing leading entries are $k_1 = 1$, $k_2 = 3$, and $k_3 = 5$, the solution set is spanned by the fundamental circuit corresponding to columns 2, 4, 5, and 6:

$$\operatorname{Ker}(\mathbf{A}) = \operatorname{Span}\left(\begin{bmatrix} -2 \\ \underline{1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ \underline{1} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ -1 \\ \underline{1} \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 1 \\ 0 \\ \underline{1} \end{bmatrix} \right) .$$ □

Exercises

6.0.8 Problem. Determine which of the following statements are true. If a statement is false, then provide a counterexample.

i. Solving a homogeneous linear system is equivalent to find the kernel of the coefficient matrix.
ii. A homogeneous linear system is never inconsistent.
iii. The only solution to a homogeneous linear system is $\vec{\mathbf{0}}$.
iv. If the coefficient matrix has more rows than columns, then the corresponding homogeneous linear system is inconsistent.
v. If the coefficient matrix has more columns than rows, then the corresponding homogeneous linear system has infinitely many solutions.
vi. A homogeneous linear system has a unique solutions if and only if the rank of the coefficient matrix equals the number of rows.

6.0.9 Problem. For which $d \in \mathbb{Q}$, does the following vector equation have infinitely many solutions?

$$x_1 \begin{bmatrix} 1 \\ -1 \\ -d \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ d-6 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

6.0.10 Problem. Solve the linear system

$$\left\{ \begin{array}{rcrcrcl} \mathrm{i}\, z_1 & + & (1+\mathrm{i})\, z_2 & & & = & 0 \\ (1-\mathrm{i})\, z_1 & + & z_2 & - & \mathrm{i}\, z_3 & = & 0 \\ & & \mathrm{i}\, z_2 & + & z_3 & = & 0 \end{array} \right\} .$$

6.1 Non-homogenous Systems

HOW SHOULD WE DESCRIBE THE SOLUTION SET OF AN ARBITRARY LINEAR SYSTEM? We start several ways to identify some consistent systems of linear equations.

6.1.0 Proposition (Universal consistency)**.** *Fix positive integers m and n. For any $(m \times n)$-matrix $\mathbf{A}$, the following are equivalent.*

a. *For all vectors $\vec{\boldsymbol{b}} \in \mathbb{K}^m$, the linear system $\mathbf{A}\,\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$ has a solution.*
b. *Each vector $\vec{\boldsymbol{b}} \in \mathbb{K}^m$ is a linear combination of the columns of $\mathbf{A}$.*
c. *The columns of the matrix $\mathbf{A}$ span $\mathbb{K}^m$.*
d. *The reduced row echelon form of $\mathbf{A}$ has a leading one in every row.*
e. *The rank of $\mathbf{A}$ equals m.*

Proof. Let $\vec{\boldsymbol{a}}_1, \vec{\boldsymbol{a}}_2, \dots, \vec{\boldsymbol{a}}_n \in \mathbb{K}^m$ denote the columns of the matrix $\mathbf{A}$.
$a \Rightarrow b$: Given a solution $\vec{\boldsymbol{v}} \in \mathbb{K}^n$ to $\mathbf{A}\,\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$, the definition of matrix multiplication implies that $v_1\,\vec{\boldsymbol{a}}_1 + v_2\,\vec{\boldsymbol{a}}_2 + \cdots + v_n\,\vec{\boldsymbol{a}}_n = \vec{\boldsymbol{b}}$, which shows that $\vec{\boldsymbol{b}}$ is a linear combination of the columns of $\mathbf{A}$.

$b \Rightarrow c$: When the vector $\vec{b} \in \mathbb{K}^m$ is a linear combination of the columns of $\mathbf{A}$, there exists scalars $v_1, v_2, \dots, v_n \in \mathbb{K}$ such that $v_1\, \vec{a}_1 + v_2\, \vec{a}_2 + \cdots + v_n\, \vec{a}_n = \vec{b}$, so the definition of a spanning set implies that $\vec{b} \in \operatorname{Span}(\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n)$.

$c \Rightarrow d$: We prove the contrapositive. If the j-th row in the reduced row echelon form of $\mathbf{A}$ does not contain a leading one, then the rightmost column of the augmented matrix $\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n & \vec{e}_j \end{bmatrix}$ contains a leading one. It follows that the linear system $\mathbf{A}\,\vec{x} = \vec{e}_j$ is inconsistent, so the columns of $\mathbf{A}$ do not span $\mathbb{K}^m$.

$d \Rightarrow a$: Assuming that the reduced row echelon form of $\mathbf{A}$ has leading one in every row, the rightmost column of the augmented matrix associated to the linear system $\mathbf{A}\,\vec{x} = \vec{b}$ cannot contain a leading one, so the equation $\mathbf{A}\,\vec{x} = \vec{b}$ is consistent.

$d \Leftrightarrow e$: By definition, the rank of the matrix $\mathbf{A}$ equals m if and only if every row of the reduced row echelon form has a leading one. □

For any consistent linear system, the solution set has an explicit description in terms of its associated homogeneous linear system.

6.1.1 Theorem (General solution sets). *Let $\mathbf{A}\,\vec{x} = \vec{b}$ be a consistent linear system and let the vector $\vec{p} \in \mathbb{K}^n$ be a particular solution. The solution set of the non-homogeneous linear system $\mathbf{A}\,\vec{x} = \vec{b}$ consists of all vectors of the form $\vec{p} + \vec{v}$ where $\vec{v}$ is any solution of the homogeneous linear system $\mathbf{A}\,\vec{x} = \vec{0}$ having the same coefficient matrix.*

The function that adds a constant vector to every vector is a ***translation***; it moves every vector by the same amount. Assuming that the non-homogeneous linear system $\mathbf{A}\,\vec{x} = \vec{b}$ is consistent, the theorem says that the solution set is obtained by translating the solution set of the homogeneous linear system $\mathbf{A}\,\vec{x} = \vec{0}$ by some particular solution $\vec{p}$.

Proof. We show that the solution set for the consistent linear system $\mathbf{A}\,\vec{x} = \vec{b}$ is equals $\{\vec{w} \in \mathbb{K}^n \mid \vec{w} := \vec{p} + \vec{v} \text{ and } \mathbf{A}\,\vec{v} = \vec{0}\}$. To establish this equality of sets, we prove containment in both directions.

$\supseteq$: Suppose that $\vec{v} \in \mathbb{K}^n$ satisfies $\mathbf{A}\,\vec{v} = \vec{0}$. The linearity of matrix multiplication [5.0.5] gives $\mathbf{A}(\vec{p} + \vec{v}) = \mathbf{A}\,\vec{p} + \mathbf{A}\,\vec{v} = \vec{b} + \vec{0} = \vec{b}$, so the vector $\vec{p} + \vec{v} \in \mathbb{K}^n$ is also a solution to $\mathbf{A}\,\vec{x} = \vec{b}$.

$\subseteq$: Suppose that $\vec{w} \in \mathbb{K}^n$ satisfies $\mathbf{A}\,\vec{w} = \vec{b}$. Set $\vec{v} := \vec{w} - \vec{p}$. The linearity of matrix multiplication [5.0.5] gives

$$\mathbf{A}\,\vec{v} = \mathbf{A}(\vec{w} - \vec{p}) = \mathbf{A}\,\vec{w} - \mathbf{A}\,\vec{p} = \vec{b} - \vec{b} = \vec{0}\,,$$

so the vector $\vec{v} \in \mathbb{K}^n$ is a solution to $\mathbf{A}\,\vec{x} = \vec{0}$. □

6.1.2 Problem. Do the planes given by the equations $x + 4y - 5z = 0$ and $2x - y + 8z = 9$ intersect? If so, describe their intersection.

Solution. The row reduction algorithm [4.2.0] gives

$$\begin{bmatrix} \underline{1} & 4 & -5 & 0 \\ 2 & -1 & 8 & 9 \end{bmatrix} \xrightarrow[\sim]{\vec{r}_2 - 2\vec{r}_1 \mapsto \vec{r}_2} \begin{bmatrix} 1 & 4 & -5 & 0 \\ 0 & \underline{-9} & 18 & 9 \end{bmatrix} \xrightarrow[\sim]{-(1/9)\vec{r}_2 \mapsto \vec{r}_2} \begin{bmatrix} 1 & 4 & -5 & 0 \\ 0 & \underline{1} & -2 & 1 \end{bmatrix} \xrightarrow[\sim]{\vec{r}_1 - 4\vec{r}_2 \mapsto \vec{r}_1} \begin{bmatrix} \underline{1} & 0 & 3 & 4 \\ 0 & \underline{1} & -2 & -1 \end{bmatrix},$$

so the solution set is $\left\{ \begin{bmatrix} 4-3t \\ -1+2t \\ t \end{bmatrix} \,\middle|\, t \in \mathbb{K} \right\} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} + \operatorname{Span}\left(\begin{bmatrix} -3 \\ 2 \\ \underline{1} \end{bmatrix} \right)$. □

6.1.3 Problem. Is the linear system $\mathbf{A}\,\vec{x} = \vec{b}$ consistent for all vectors $\vec{b} \in \mathbb{K}^3$ where

$$\mathbf{A} := \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}?$$

Solution. The row reduction algorithm [4.2.0] gives

$$\left[\begin{array}{rrrr} \underline{1} & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{array}\right] \xrightarrow[\sim]{\substack{\vec{r}_2 + 4\vec{r}_1 \mapsto \vec{r}_2 \\ \vec{r}_3 + 3\vec{r}_1 \mapsto \vec{r}_3}} \left[\begin{array}{rrrr} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & \underline{7} & 5 & b_3 + 3b_1 \end{array}\right] \xrightarrow[\sim]{\vec{r}_2 - 2\vec{r}_3 \mapsto \vec{r}_2} \left[\begin{array}{rrrr} 1 & 3 & 4 & b_1 \\ 0 & 0 & 0 & \underline{b_2 - 2b_3 - 2b_1} \\ 0 & 7 & 5 & b_3 + 3b_1 \end{array}\right].$$

Since $-2b_1 + b_2 - 2b_3$ is generally nonzero, the non-homogeneous linear system $\mathbf{A}\,\vec{x} = \vec{0}$ is not consistent for all $\vec{b} \in \mathbb{K}^3$. □

Exercises

6.1.4 Problem. Determine which of the following statements are true. If a statement is false, then provide a counterexample.

i. A linear system is consistent if and only if the columns of its coefficient matrix span the ambient coordinate space.

ii. If the rank of the matrix $\mathbf{A}$ is less than the number of rows, then the linear system $\mathbf{A}\mathbf{x} = \vec{b}$ is never consistent.

iii. The reduced row echelon form of a matrix must have a leading entry in every row.

iv. The solution set to any linear system is a translation of the kernel of its coefficient matrix.

v. The reduced row echelon form of the augmented matrix associated to a non-homogeneous linear system includes a particular solution.

6.1.5 Problem. The ***Gell–Mann matrices*** are the following complex (3×3)-matrices:

$$\mathbf{G}_1 := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{G}_2 := \begin{bmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{G}_3 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{G}_4 := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{G}_5 := \begin{bmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{bmatrix} \quad \mathbf{G}_6 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \mathbf{G}_7 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{bmatrix} \quad \mathbf{G}_8 := \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

i. How many solutions does the matrix equation

$$x_1\,\mathbf{G}_1 + x_2\,\mathbf{G}_2 + x_3\,\mathbf{G}_3 + x_4\,\mathbf{G}_4 + x_5\,\mathbf{G}_5 + x_6\,\mathbf{G}_6 + x_7\,\mathbf{G}_7 + x_8\,\mathbf{G}_8 = \mathbf{0}$$

have?

ii. Describe $\operatorname{Span}(\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3, \mathbf{G}_4, \mathbf{G}_5, \mathbf{G}_6, \mathbf{G}_7, \mathbf{G}_8)$.