Exercises

2.2.4 Problem. Determine which of the following statements are true. If a statement is false, then provide a counterexample.

i. The zero vector space has no basis.
ii. Every vector space has a finite basis.
iii. Every vector space that is spanned by a finite set of vectors has a basis.
iv. Every linear subspace of a finite-dimensional vector space is finite-dimensional.
v. For any nonnegative integer n, there exists a vector space of dimension n.

2.3 Coordinates

WHY ARE BASES IMPORTANT FOR COMPUTATIONS? Choosing a basis for a $\mathbb{K}$-vector space V allows one to identify V with a coordinate space $\mathbb{K}^n$ and, thereby, exploit our many computation techniques.

2.3.0 Proposition (Basis means unique linear combination)**.** *The vectors $\boldsymbol{v}_1, \boldsymbol{v}_2 \dots, \boldsymbol{v}_n$ form a basis for the $\mathbb{K}$-vector space V if and only if any vector $\boldsymbol{w}$ in V is a unique linear combination of the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$.*

Proof.

$\Rightarrow$: Suppose that the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$ form a basis for V. Fix a vector $\boldsymbol{w}$ in V. Since $V = \operatorname{Span}(\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n)$, there exists scalars $c_1, c_2, \dots, c_n$ in $\mathbb{K}$ such that $\boldsymbol{w} = c_1\, \boldsymbol{v}_1 + c_2\, \boldsymbol{v}_2 + \cdots + c_n\, \boldsymbol{v}_n$. If we also have $\boldsymbol{w} = d_1\, \boldsymbol{v}_1 + d_2\, \boldsymbol{v}_2 + \cdots + d_n\, \boldsymbol{v}_n$ for some scalars $d_1, d_2, \dots, d_n$ in $\mathbb{K}$, then we obtain

$$\boldsymbol{0} = \boldsymbol{w} - \boldsymbol{w} = (c_1 - d_1)\, \boldsymbol{v}_1 + (c_2 - d_2)\, \boldsymbol{v}_2 + \cdots + (c_n - d_n)\, \boldsymbol{v}_n\,.$$

Since the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$ are linearly independent, it follows that $c_1 - d_1 = c_2 - d_2 = \cdots = c_n - d_n = 0$ and the two linear combinations are the same.

$\Leftarrow$: Suppose that each vector $\boldsymbol{w}$ in V is a unique linear combination of the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$. Because this holds for every vector in V, it follows that $\operatorname{Span}(\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n) = V$. The definition of linear independence is equivalent by saying that the zero vector $\boldsymbol{0}$ is a unique linear combination of the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$. Thus, the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$ form a basis for V $\square$

By focusing on the coefficients in the unique linear combination of the basis vectors, we obtain the associated coordinate vector.

2.3.1 Definition. Let $\mathcal{B} := (\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n)$ be an ordered basis for the $\mathbb{K}$-vector space V. For any vector $\boldsymbol{w}$ in V, the ***coordinate vector*** of $\boldsymbol{w}$ relative to the basis $\mathcal{B}$ is the unique vector $(w)_{\mathcal{B}} := [c_1 \;\; c_2 \;\; \cdots \;\; c_n]^{\mathsf{T}}$ in $\mathbb{K}^n$ such that $\boldsymbol{w} = c_1\, \boldsymbol{v}_1 + c_2\, \boldsymbol{v}_2 + \cdots + c_n\, \boldsymbol{v}_n$.

An ordered basis is also called a ***frame***.

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2.3.2 Problem. Consider $V := \{f \in \mathbb{R}[t]_{\leqslant 2} \mid \int_0^1 f(t)\,dt = 0\}$. Show that V is an $\mathbb{R}$-vector space and $\mathcal{B} := (t - 1/2, t^2 - 1/3)$ is an ordered basis. Compute the coordinate vector of $3t^2 - 2t$ with relative to $\mathcal{B}$.

Solution. The zero polynomial belongs to V, so we have $V \neq \varnothing$. For any polynomials f, g in V and any scalars $b, c \in \mathbb{R}$, we have

$$\int_0^1 (b\,f + c\,g)(t)\,dt = b\left(\int_0^1 f(t)\,dt\right) + c\left(\int_0^1 g(t)\,dt\right) = b(0) + c(0) = 0\,,$$

and the linear combination $b\,f + c\,g$ also lies in V. Thus, the subspace test [1.2.0] shows that V is a linear subspace of $\mathbb{R}[t]$.

Since any vector f in $\mathbb{R}[t]_{\leqslant 2}$ has the form $f := a_0 + a_1\,t + a_2\,t^2$ for some scalars $a_0, a_1, a_2 \in \mathbb{R}$, it follows that

$$0 = \int_0^1 f(t)\,dt = \int_0^1 a_0 + a_1\,t + a_2\,t^2\,dt = \Big[a_0\,t + \tfrac{a_1}{2}\,t^2 + \tfrac{a_2}{3}\,t^3\Big]_0^1 = a_0 + \tfrac{1}{2}a_1 + \tfrac{1}{3}a_2\,.$$

As $a_0 = -a_1/2 - a_2/3$, we see that $f = a_1(t - 1/2) + a_2(t^2 - 1/3)$ and $V = \operatorname{Span}(t - 1/2, t^2 - 1/3)$. To show that these functions are linearly independent, consider a linear relation $c_1(t - 1/2) + c_2(t^2 - 1/3) = 0$. Evaluating at $t = 1/2$ and $t = 1/\sqrt{3}$ implies that $c_1 = c_2 = 0$. We conclude that $\mathcal{B} = (t - 1/2, t^2 - 1/3)$ is an ordered basis for $\mathbb{R}$-vector space V and $\dim(V) = 2$.

Finally, the equation $3t^2 - 2t = -2\,(t - 1/2) + 3\,(t^2 - 1/3)$ implies that the coordinate vector is $(3t^2 - 2t)_{\mathcal{B}} = [-2 \;\; 3]^{\mathsf{T}} \in \mathbb{R}^2$. □

2.3.3 Definition. For all nonnegative integers n and k, the binomial coefficient $\binom{n}{k}$ is the number of subsets of the set $\{1, 2, \dots, n\}$ with k elements. For instance, the 2-element subsets of $\{1, 2, 3, 4\}$ are $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$, and $\{3, 4\}$, so $\binom{4}{2} = 6$.

Some values are easy to determine.

- For any nonnegative integer n, we have $\binom{n}{0} = 1$ because the empty set is the unique set with no elements.
- For any nonnegative integer n, we have $\binom{n}{n} = 1$ because the set $\{1, 2, \dots, n\}$ itself is the unique set with n elements.
- For any nonnegative integer n, we have $\binom{n}{2} = n(n-1)/2$ because there are n ways to choose the first element, $n - 1$ ways to choose a different second element, and 2 ways to order them.

2.3.4 Binomial Theorem. *For all nonnegative integers n and all scalars a in $\mathbb{K}$, the coordinate vector of the polynomial $(t + a)^n$ in $\mathbb{K}[t]_{\leqslant n}$ relative to the monomial basis $\mathcal{M} := (1, t, \dots, t^n)$ is*

$$((t + a)^n)_{\mathcal{M}} = \begin{bmatrix} \binom{n}{0}a^n & \binom{n}{1}a^{n-1} & \binom{n}{2}a^{n-2} & \cdots & \binom{n}{n}a^0 \end{bmatrix}^{\mathsf{T}} \in \mathbb{K}^{n+1}\,.$$

In other words, we have $(t + a)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}\,t^k$.

Proof. When we expand $(t + a)^n = (t + a)(t + a)\cdots(t + a)$ using the distributive property, every term is a product of n factors and each factor is either t or a. The number of terms with k factors of t and $n - k$ factors of a is the coefficient of $t^k a^{n-k}$. This is exactly the number of ways to choose k of the n binomials that will contribute a t, so

$$(t + a)^n = \binom{n}{0}\,a^n + \binom{n}{1}\,a^{n-1}\,t + \binom{n}{2}\,a^{n-2}\,t^2 + \cdots + \binom{n}{n}a^0\,t^n = \sum_{k=0}^{n}\binom{n}{k}\,a^{n-k}\,t^k\,,$$

and $((t + a)^n)_{\mathcal{M}} = \big[\binom{n}{0}a^n \;\; \binom{n}{1}a^{n-1} \;\; \binom{n}{2}a^{n-2} \;\; \cdots \;\; \binom{n}{n}a^0\big]^{\mathsf{T}}$. □

2.3.5 Problem. Let n be a nonnegative integer and let a be scalar in $\mathbb{K}$. Show that that the list of polynomials $\big(1, (t-a), (t-a)^2, \dots, (t-a)^n\big)$ is an ordered basis for the $\mathbb{K}$-vector space $\mathbb{K}[t]_{\leqslant n}$.

Solution. The $n+1$ entries in the given list lie in $\mathbb{K}[t]_{\leqslant n}$. Knowing that $\dim \mathbb{K}[t]_{\leqslant n} = n+1$, it suffices to prove that these polynomials span $\mathbb{K}[t]_{\leqslant n}$. For all $0 \leqslant k \leqslant n$, the Binomial Theorem shows that

$$t^k = \big((t-a)+a\big)^k = \sum_{j=0}^{k} \binom{k}{j} a^{k-j}\,(t-a)^j \,,$$

so the monomial t^k lies in $\operatorname{Span}(1, (t-a), (t-a)^2, \dots, (t-a)^n)$. Since the canonical basis for $\mathbb{K}[t]_{\leqslant n}$ is $(1, t, t^2, \dots, t^n)$, we deduce that $\mathbb{K}[t]_{\leqslant n} \subseteq \operatorname{Span}\big(1, (t-a), (t-a)^2, (t-a)^3, \dots, (t-a)^n\big) \subseteq \mathbb{K}[t]_{\leqslant n}$. □

Exercises

2.3.6 Problem. Determine which of the following statements are true. If a statement is false, then provide a counterexample.

i. The set $\mathbb{R}^{\mathbb{R}}$ of a real-valued functions on the real line is a $\mathbb{R}$-vector space.
ii. The set $\mathbb{Z}^{\mathbb{N}}$ of a integer sequences is a $\mathbb{Q}$-vector space.
iii. The set $\mathbb{Q}[t]$ of rational polynomials is a linear subspace of $\mathbb{Q}^{\mathbb{C}}$.

2.3.7 Problem. Fix a nonnegative integer n. For each $0 \leqslant k \leqslant n$, consider the ***Bernstein polynomial***

$$\mathrm{b}_{k,n}(t) \coloneqq \binom{n}{k}\, t^k (1-t)^{n-k} \in \mathbb{Q}[t] \,.$$

i. Show that $\mathrm{b}_{0,n}(t), \mathrm{b}_{1,n}(t), \dots, \mathrm{b}_{n,n}(t)$ form a basis for $\mathbb{Q}[t]_{\leqslant n}$.
ii. Prove that $\displaystyle\sum_{j=0}^{n} \mathrm{b}_{j,n}(t) = 1$.

3
Linear Transformations

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Maps between vectors spaces are as crucial as the spaces themselves. This chapter illustrates the prevalence and the significance of maps that are compatible with taking linear combinations.

3.0 Homomorphisms

What are the most important maps between vector spaces? When studying a particular type of mathematical object, the maps that preserve the underlying structure are especially important.

3.0.0 Definition. Let V and W be two $\mathbb{K}$-vector spaces. A ***linear map*** (also known as a ***linear transformation*** or ***homomorphism***) is a map $T \colon V \to W$ such that, for all vectors $\boldsymbol{v}, \boldsymbol{w}$ in V and all scalars b, c in $\mathbb{K}$, we have $T[b\,\boldsymbol{v} + c\,\boldsymbol{w}] = b\,T[\boldsymbol{v}] + c\,T[\boldsymbol{w}]$. The set of all linear maps from V to W is denoted by $\operatorname{Hom}(V, W)$.

The word "homomorphism" comes from ancient Greek: $\grave{o}\mu\acute{o}\varsigma$ (homos) means "same" and $\mu o\rho\phi\acute{\eta}$ (morphe) means "form" or "shape".

3.0.1 Problem. Show that left multiplication by a fixed $(m \times n)$-matrix defines a linear map from $\mathbb{K}^n$ to $\mathbb{K}^m$.

Solution. Given an $(m \times n)$-matrix $\mathbf{A}$, consider the map defined, for all vectors $\boldsymbol{v}$ in $\mathbb{K}^n$, by $\boldsymbol{v} \mapsto \mathbf{A}\,\boldsymbol{v}$. For all vectors $\boldsymbol{v}, \boldsymbol{w}$ in $\mathbb{K}^n$ and all scalars b, c in $\mathbb{K}$, the properties of the matrix multiplication establish that $\mathbf{A}(b\,\boldsymbol{v} + c\,\boldsymbol{w}) = b(\mathbf{A}\,\boldsymbol{v}) + c(\mathbf{A}\,\boldsymbol{w})$. □

3.0.2 Problem. Prove that multiplication by a fixed polynomial defines a homomorphism from $\mathbb{K}[t]$ to itself.

Calculus shows that differentiation defines a linear map from the vector space of all differentiation functions to the vector space of all functions. Similarly, integration defines a linear map from the vector space of all integrable functions to the vector space of all functions.

Solution. Given a polynomial h in $\mathbb{K}[t]$, consider the map defined, for all $f \in \mathbb{K}[t]$, by $f \mapsto f\,h$. For all polynomials f, g in $\mathbb{K}[t]$ and all scalars b, c in $\mathbb{K}$, the distributivity of multiplication implies that $(b\,f + c\,g)\,h = b(f\,h) + c(g\,h)$. □

3.0.3 Proposition (Properties of linear maps)**.** *Let V and W be two $\mathbb{K}$-vector spaces.*

i. *For any linear map $T \colon V \to W$, we have $T[\mathbf{0}_V] = \mathbf{0}_W$.*
ii. *The zero map $0 \colon V \to W$, that sends each vector in V to the additive identity in W, and the identity map $\mathrm{id}_V \colon V \to V$ are both linear.*
iii. *The composition of linear maps is again linear.*
iv. *A linear combination of linear maps is linear.*

Proof.

i. The definition of a linear map and the additive identity property in V give $T[\mathbf{0}_V] = T[\mathbf{0}_V + \mathbf{0}_V] = T[\mathbf{0}_V] + T[\mathbf{0}_V]$. Using the additive identity and additive inverse properties in W, we obtain

$$\mathbf{0}_W = T[\mathbf{0}_V] - T[\mathbf{0}_V] = T[\mathbf{0}_V] + T[\mathbf{0}_V] - T[\mathbf{0}_V] = T[\mathbf{0}_V] + \mathbf{0}_W = T[\mathbf{0}_V]\,.$$

ii. For all vectors $\boldsymbol{v}, \boldsymbol{w}$ in V and all scalars b, c in $\mathbb{K}$, we have

$$\begin{aligned} 0[b\,\boldsymbol{v} + c\,\boldsymbol{w}] &= \mathbf{0}_W = b\,\mathbf{0}_W + c\,\mathbf{0}_W = b\,0[\boldsymbol{v}] + c\,0[\boldsymbol{w}]\,, \\ \mathrm{id}_V[b\,\boldsymbol{v} + c\,\boldsymbol{w}] &= b\,\boldsymbol{v} + c\,\boldsymbol{w} = b\;\mathrm{id}_V[\boldsymbol{v}] + c\;\mathrm{id}_V[\boldsymbol{w}]\,. \end{aligned}$$

iii. Let $S \colon U \to V$ and $T \colon V \to W$ be linear maps between $\mathbb{K}$-vector spaces. For all vectors $\boldsymbol{v}, \boldsymbol{w}$ in U and all scalars b, c in $\mathbb{K}$, we have

$$\begin{aligned} (T \circ S)[b\,\boldsymbol{v} + c\,\boldsymbol{w}] &= T\big[S[b\,\boldsymbol{v} + c\,\boldsymbol{w}]\big] \\ &= T\big[b\,S[\boldsymbol{v}] + c\,S[\boldsymbol{w}]\big] \\ &= b\,T\big[S[\boldsymbol{v}]\big] + c\,T\big[S[\boldsymbol{w}]\big] = b\,(T \circ S)[\boldsymbol{v}] + c\,(T \circ S)[\boldsymbol{w}]\,. \end{aligned}$$

iv. Let $T \colon V \to W$ and $T' \colon V \to W$ be linear maps between $\mathbb{K}$-vector spaces. For all vectors $\boldsymbol{v}, \boldsymbol{w}$ in V and all scalars a, b, c, d in $\mathbb{K}$, we have

$$\begin{aligned} (a\,T + b\,T')[c\,\boldsymbol{v} + d\,\boldsymbol{w}] &= a\,T[c\,\boldsymbol{v} + d\,\boldsymbol{w}] + b\,T'[c\,\boldsymbol{v} + d\,\boldsymbol{w}] \\ &= ac\,T[\boldsymbol{v}] + ad\,T[\boldsymbol{w}] + bc\,T'[\boldsymbol{v}] + bd\,T'[\boldsymbol{v}] \\ &= c\big(a\,T[\boldsymbol{v}] + b\,T'[\boldsymbol{v}]\big) + d\big(a\,T[\boldsymbol{w}] + b\,T'[\boldsymbol{w}]\big) \\ &= c\,(a\,T + b\,T')[\boldsymbol{v}] + d\,(a\,T + b\,T')[\boldsymbol{w}]\,. \qquad \square \end{aligned}$$

3.0.4 Corollary. *For all $\mathbb{K}$-vector spaces V and W, the set* $\mathrm{Hom}(V, W)$ *of linear maps is a linear subspace of the $\mathbb{K}$-vector space V^W.*

Proof. By combining second and fourth properties of linear maps, the subspace test [1.2.0] shows that $\mathrm{Hom}(V, W)$, equipped with pointwise operations, is a linear subspace of V^W. $\square$

3.0.5 Problem. Show that the functions $\cos(x), \sin(x)$ are not linear.

Solution. Since $\cos(0) = 1 \neq 0$, the cosine function cannot be linear. Since $\sin\big(\frac{\pi}{2} - \frac{\pi}{3}\big) = \sin\big(\frac{\pi}{6}\big) = \frac{1}{2}$ and $\sin\big(\frac{\pi}{2}\big) - \sin\big(\frac{\pi}{3}\big) = 1 - \frac{\sqrt{3}}{2} \neq \frac{1}{2}$, the sine function is not linear. $\square$

3.0.6 Problem. Let V and W be $\mathbb{K}$-vector spaces. Show evaluation at a fixed vector $\boldsymbol{v}$ in V defines a linear map from $\mathrm{Hom}(V, W)$ to W.

Solution. For all linear maps f, g in $\mathrm{Hom}(V, W)$ and all scalars b, c in $\mathbb{K}$, we have $(b\,f + c\,g)[\boldsymbol{v}] = b(f[\boldsymbol{v}]) + c(g[\boldsymbol{v}])$ because both the addition and scalar multiplication on $\mathrm{Hom}(V, W)$ are defined pointwise. $\square$

3.0.7 Proposition (Linear maps via a basis)**.** *A linear map is uniquely determined by its values on a basis and may take arbitrary values on a basis.*

Proof. Consider a linear map $T\colon V \to W$ and a basis $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ of the $\mathbb{K}$-vector space V. For all $1 \leqslant k \leqslant n$, set $\boldsymbol{w}_k := T[\boldsymbol{v}_k]$. For any vector $\boldsymbol{u}$ in V, there exists unique scalars $c_1, c_2, \ldots, c_n$ in $\mathbb{K}$ such that $\boldsymbol{u} = c_1\,\boldsymbol{v}_2 + c_2\,\boldsymbol{v}_2 + \cdots + c_n\,\boldsymbol{v}_n$, because $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ is a basis for V. The linearity of the map T implies that

$$\begin{aligned} T[\boldsymbol{u}] &= T[c_1\,\boldsymbol{v}_1 + c_2\,\boldsymbol{v}_2 + \cdots + c_n\,\boldsymbol{v}_n] \\ &= c_1(T[\boldsymbol{v}_1]) + c_2(T[\boldsymbol{v}_2]) + \cdots + c_n(T[\boldsymbol{v}_n]) \\ &= c_1\,\boldsymbol{w}_1 + c_2\,\boldsymbol{w}_2 + \cdots + c_n\,\boldsymbol{w}_n\,, \end{aligned}$$

so the output $T[\boldsymbol{u}]$ is determined by the vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n$.

Given an arbitrary collection of vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n$ in W, define the map $T'\colon V \to W$ by $T'[\boldsymbol{u}] := c_1\,\boldsymbol{w}_1 + c_2\,\boldsymbol{w}_2 + \cdots + c_n\,\boldsymbol{w}_n$. For any vector $\boldsymbol{u}' := c_1'\,\boldsymbol{v}_1 + c_2'\,\boldsymbol{v}_2 + \cdots + c_n'\,\boldsymbol{v}_n$ in V and all scalars a, b in $\mathbb{K}$, we obtain

$$\begin{aligned} T'[a\boldsymbol{u} + b\boldsymbol{u}'] &= T[(ac_1 + bc_1')\,\boldsymbol{v}_1 + (ac_2 + bc_2')\,\boldsymbol{v}_2 + \cdots + (ac_n + bc_n')\,\boldsymbol{v}_n] \\ &= (ac_1 + bc_1')\,\boldsymbol{w}_1 + (ac_2 + bc_2')\,\boldsymbol{w}_2 + \cdots + (ac_n + bc_n')\,\boldsymbol{w}_n \\ &= a(c_1\,\boldsymbol{w}_1 + c_2\,\boldsymbol{w}_2 + \cdots + c_n\,\boldsymbol{w}_n) + b(c_1'\,\boldsymbol{w}_1 + c_2'\,\boldsymbol{w}_2 + \cdots + a_n'\,\boldsymbol{w}_n) \\ &= a(T'[\boldsymbol{u}]) + b(T'[\boldsymbol{u}'])\,. \end{aligned}$$

Thus, T' is a linear map satisfying $T'[\boldsymbol{v}_k] = \boldsymbol{w}_k$ for all $1 \leqslant k \leqslant n$. □

Exercises

3.0.8 Problem. Determine which of the following statements are true. If a statement is false, then provide a counterexample.

i. The function $f\colon \mathbb{K} \to \mathbb{K}$ defined by $f(x) = x + 1$ is a linear transformation.

ii. There exists at least one linear transformation between any two $\mathbb{K}$-vector spaces.

iii. Conjugation of complex numbers defines an $\mathbb{R}$-linear map from $\mathbb{C}$ to itself, but not a $\mathbb{C}$-linear map.

3.1 Kernels and Images

WHAT ARE THE CANONICAL LINEAR SUBSPACES ASSOCIATED TO A LINEAR TRANSFORMATION? Extending our nomenclature for matrices, any linear map determines the two fundamental linear subspaces.

3.1.0 Definition. For a linear map $T\colon V \to W$, the ***kernel*** and ***image*** are the following subsets:

(kernel) $\operatorname{Ker}(T) := \{\boldsymbol{v} \in V \mid T[\boldsymbol{v}] = \boldsymbol{0}\}$,

(image) $\operatorname{Im}(T) := \{\boldsymbol{w} \in W \mid \text{there exists } \boldsymbol{v} \in V \text{ such that } \boldsymbol{w} = T[\boldsymbol{v}]\}$.

3.1.1 Remark. When $\mathbf{A}$ is an $(m \times n)$-matrix and $T\colon \mathbb{K}^n \to \mathbb{K}^m$ is defined by $T[\vec{x}] := \mathbf{A}\,\vec{x}$, the kernel $\operatorname{Ker}(T)$ consists of the solutions to homogeneous linear system $T[\vec{x}] = \mathbf{A}\vec{x} = \vec{0}$.

3.1.2 Proposition. *For any linear map $T\colon V \to W$, the subsets* $\operatorname{Ker}(T)$ *and* $\operatorname{Im}(T)$ *are linear subspaces of V and W respectively.*

Proof. The properties [3.0.3] of linear maps include $T[\mathbf{0}_V] = \mathbf{0}_W$, so $\operatorname{Ker}(T) \neq \varnothing$ and $\operatorname{Im}(T) \neq \varnothing$. For all vectors $\boldsymbol{v}, \boldsymbol{v}'$ in $\operatorname{Ker}(T)$ and all scalars b, c in $\mathbb{K}$, we have

$$T[b\,\boldsymbol{v} + c\,\boldsymbol{v}'] = b(T[\boldsymbol{v}]) + c(T[\boldsymbol{v}']) = b\,\mathbf{0}_W + c\,\mathbf{0}_W = \mathbf{0}_W\,,$$

so the linear combination $b\,\boldsymbol{v} + c\,\boldsymbol{v}$ lies in $\operatorname{Ker}(T)$. Furthermore, for all vectors $\boldsymbol{w}, \boldsymbol{w}'$ in $\operatorname{Im}(T)$, there exists vectors $\boldsymbol{v}, \boldsymbol{v}'$ in V such that $T[\boldsymbol{v}] = \boldsymbol{w}$ and $T[\boldsymbol{v}'] = \boldsymbol{w}'$. Hence, for all scalars b, c in $\mathbb{K}$, we obtain $T[b\,\boldsymbol{v} + c\,\boldsymbol{v}'] = b(T[\boldsymbol{v}]) + c(T[\boldsymbol{v}']) = b\,\boldsymbol{w} + c\,\boldsymbol{w}' \in \operatorname{Im}(T)$. Thus, the subspace test [1.2.0] shows that $\operatorname{Ker}(T)$ is a linear subspace of V and that $\operatorname{Im}(T)$ is a linear subspace of W. □

3.1.3 Definition. A map $T\colon V \to W$ is defined to be
- ***injective*** if, for all vectors $\boldsymbol{v}, \boldsymbol{w}$ in V, the equality $T[\boldsymbol{v}] = T[\boldsymbol{w}]$ implies that $\boldsymbol{v} = \boldsymbol{w}$.
- ***surjective*** if, for any vector $\boldsymbol{w}$ in W, there exists a vector $\boldsymbol{v}$ in V such that $T[\boldsymbol{v}] = \boldsymbol{w}$.

3.1.4 Proposition (Injectivity and surjectivity via linearity)**.**
i. *A linear map $T\colon V \to W$ is injective if and only if* $\operatorname{Ker}(T) = \{\mathbf{0}_V\}$.
ii. *A linear map $T\colon V \to W$ is surjective if and only if* $\operatorname{Im}(T) = W$.

Proof.
i. $\Rightarrow$: Suppose that T is injective. For any vector $\boldsymbol{v}$ in $\operatorname{Ker}(T)$, the properties [3.0.3] of linear maps imply that $T[\boldsymbol{v}] = \mathbf{0} = T[\mathbf{0}]$. Since T is injective, we have $\boldsymbol{v} = \mathbf{0}$, which establishes that $\operatorname{Ker}(T) = \{\mathbf{0}\}$.
$\Leftarrow$: Suppose that $\operatorname{Ker}(T) = \{\mathbf{0}_V\}$. For any vectors $\boldsymbol{v}, \boldsymbol{w}$ in V such that$T[\boldsymbol{v}] = T[\boldsymbol{w}]$, we have $\mathbf{0}_W = T[\boldsymbol{v}] - T[\boldsymbol{w}] = T[\boldsymbol{v} - \boldsymbol{w}]$, which means $\boldsymbol{v} - \boldsymbol{w} \in \operatorname{Ker}(T)$. It follows that $\boldsymbol{v} - \boldsymbol{w} = \mathbf{0}_V$ and $\boldsymbol{v} = \boldsymbol{w}$, establishing that T is injective.
ii. The map T is surjective if and only if, for any $\boldsymbol{w} \in W$, there exists $\boldsymbol{v} \in V$ such that $T[\boldsymbol{v}] = \boldsymbol{w}$, which is equivalent to $\operatorname{Im}(T) = V$. □

3.1.5 Problem. Consider the linear map $S\colon C(\mathbb{R}) \to C^1(\mathbb{R})$ defined by $(S[f])(x) := \int_0^x f(t)\,dt$. Determine whether S is injective or surjective.

Solution. When $0 = S[f] = \int_0^x f(t)\,dt$, the Fundamental Theorem of Calculus implies that $0 = \frac{d}{dx}\int_0^x f(t)\,dt = f(x)$, so $\operatorname{Ker}(S) = \{0\}$ and S is injective. Since $(S[f])(0) = \int_0^0 f(t)\,dt = 0$ and $e^0 = 1$, there does not exists a function $f \in C(\mathbb{R})$ such that $S[f] = e^x$, which shows that S is not surjective. The image $\operatorname{Im}(S)$ is the set of functions $g \in C^1(\mathbb{R})$ satisfying $g(0) = 0$, because the Fundamental Theorem of Calculus also establishes that $(S[g'])(x) = g(x) - g(0)$. □

3.1.6 Theorem (Dimension formula)**.** *Let V and W be $\mathbb{K}$-vector spaces such that V has a finite dimension. For any linear map $T\colon V \to W$, we have*

$$\dim(V) = \dim\bigl(\operatorname{Ker}(T)\bigr) + \dim\bigl(\operatorname{Im}(T)\bigr)\,.$$

Proof. Set $n := \dim(V)$. Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k$ be a basis for $\operatorname{Ker}(T)$, so that $k = \dim\bigl(\operatorname{Ker}(T)\bigr)$. Since V is finite-dimensional, the extremal properties of a basis [2.2.1] show that we can extend this linearly independent set to a basis of V: there exists vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_{n-k}$ in V such that $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k, \boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_{n-k}$ is a basis of V. For all indices j satisfying $1 \leqslant j \leqslant n-k$, set $\boldsymbol{w}_j := T[\boldsymbol{v}_j]$. We claim that the vectors $\boldsymbol{w}_1, \boldsymbol{w}_2 \dots, \boldsymbol{w}_{n-k}$ form a basis for $\operatorname{Im}(T)$. This claim implies that the image $\operatorname{Im}(T)$ has dimension $n-k = \dim(V) - \dim\bigl(\operatorname{Ker}(T)\bigr)$ and confirms the dimension formula.

To prove the claim, we first show $\operatorname{Im}(T) = \operatorname{Span}(\boldsymbol{w}_1, \boldsymbol{w}_2 \dots, \boldsymbol{w}_{n-k})$. The definition of the image implies that, for any vector $\boldsymbol{w}$ in $\operatorname{Im}(T)$, there exists a vector $\boldsymbol{v}$ in V such that $\boldsymbol{w} = T[\boldsymbol{v}]$. Given our chosen basis for V, there exists scalars $a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_{n-k}$ in $\mathbb{K}$ such that $\boldsymbol{v} = a_1\,\boldsymbol{u}_1 + a_2\,\boldsymbol{u}_2 + \cdots + a_k\,\boldsymbol{u}_k + b_1\,\boldsymbol{v}_1 + b_2\,\boldsymbol{v}_2 + \cdots + b_{n-k}\,\boldsymbol{v}_{n-k}$. Applying the linear map T, we obtain

$$\begin{aligned}
\boldsymbol{w} &= T[\boldsymbol{v}] \\
&= T[a_1\,\boldsymbol{u}_1 + a_2\,\boldsymbol{u}_2 + \cdots + a_k\,\boldsymbol{u}_k + b_1\,\boldsymbol{v}_1 + b_2\,\boldsymbol{v}_2 + \cdots + b_{n-k}\,\boldsymbol{v}_{n-k}] \\
&= a_1(T[\boldsymbol{u}_1]) + a_2(T[\boldsymbol{u}_2]) + \cdots + a_k(T[\boldsymbol{u}_k]) \\
&\qquad + b_1(T[\boldsymbol{v}_1]) + b_2(T[\boldsymbol{v}_2]) + \cdots + b_{n-k}(T[\boldsymbol{v}_{n-k}]) \\
&= a_1\,\boldsymbol{0} + a_2\,\boldsymbol{0} + \cdots + a_k\,\boldsymbol{0} + b_1\,\boldsymbol{w}_1 + b_2\boldsymbol{w}_2 + \cdots + b_{n-k}\,\boldsymbol{w}_{n-k} \\
&= b_1\,\boldsymbol{w}_1 + b_2\boldsymbol{w}_2 + \cdots + b_{n-k}\,\boldsymbol{w}_{n-k}\,,
\end{aligned}$$

so $\boldsymbol{w} \in \operatorname{Span}(\boldsymbol{w}_1, \boldsymbol{w}_2 \dots, \boldsymbol{w}_{n-k})$ and $\operatorname{Im}(T) = \operatorname{Span}(\boldsymbol{w}_1, \boldsymbol{w}_2 \dots, \boldsymbol{w}_{n-k})$.

To establish linear independence, suppose that there exists scalars $c_1, c_2, \dots, c_{n-k}$ in $\mathbb{K}$ such that $c_1\,\boldsymbol{w}_1 + c_2\,\boldsymbol{w}_2 + \cdots + c_{n-k}\,\boldsymbol{w}_{n-k} = \boldsymbol{0}$. It follows that, for the vector $\boldsymbol{u} := c_1\,\boldsymbol{v}_1 + c_2\,\boldsymbol{v}_2 + \cdots + c_{n-k}\,\boldsymbol{v}_{n-k}$, we have

$$\begin{aligned}
T[\boldsymbol{u}] &= c_1(T[\boldsymbol{v}_1]) + c_2(T[\boldsymbol{v}_2]) + \cdots + c_{n-k}(T[\boldsymbol{v}_{n-k}]) \\
&= c_1\,\boldsymbol{w}_1 + c_2\,\boldsymbol{w}_2 + \cdots + c_{n-k}\,\boldsymbol{w}_{n-k} = \boldsymbol{0}\,,
\end{aligned}$$

so $\boldsymbol{u} \in \operatorname{Ker}(T)$. Since the vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n$ span $\operatorname{Ker}(T)$, there exists scalars $d_1, d_2, \dots, d_k \in \mathbb{K}$ such that $\boldsymbol{u} = d_1\,\boldsymbol{u}_1 + d_2\,\boldsymbol{u}_2 + \cdots + d_k\,\boldsymbol{u}_k$, which implies that

$$\boldsymbol{0} = \boldsymbol{u} - \boldsymbol{u} = d_1\,\boldsymbol{u}_1 + d_2\,\boldsymbol{u}_2 + \cdots + d_k\,\boldsymbol{u}_k - c_1\,\boldsymbol{v}_1 - c_2\,\boldsymbol{v}_2 - \cdots - c_{n-k}\,\boldsymbol{v}_{n-k}\,.$$

Hence, we see that $d_1 = d_2 = \ldots = d_n = c_1 = c_2 = \cdots = c_{n-k} = 0$, because the vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k, \boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_{n-k}$ form a basis for V. Therefore, the vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_{n-k}$ are linearly independent and form a basis for $\operatorname{Im}(T)$. $\square$