

Exercises

3.1.7 Problem. Determine which of the following statements are true. If a statement is false, then provide a counterexample.

i. The kernel of a linear map always contains the additive identity from is domain.
ii. The image of a linear map may be the empty set.
iii. The zero homomorphism is never injective.
iv. The zero homomorphism is surjective if and only if the target vector space is the zero space.
v. The identity map is always bijective.

3.1.8 Problem. The set of all traceless $(n \times n)$-matrices,

$$\mathfrak{sl}(n, \mathbb{C}) := \{\mathbf{A} \in \mathbb{C}^{n \times n} \mid \operatorname{tr}(\mathbf{A}) = 0\} \,,$$

is a linear subspace. Find a basis for $\mathfrak{sl}(n, \mathbb{C})$. What is the dimension of $\mathfrak{sl}(n, \mathbb{C})$?

3.2 Invertible Linear maps

How can a linear map have an inverse? We first record some properties for the composition of linear maps.

3.2.0 Remark. For any two linear maps $S \colon U \to V$ and $T \colon V \to W$, the ***product*** $T\,S \colon U \to W$ is the linear map defined, for all $\boldsymbol{u} \in U$, by $(T\,S)[\boldsymbol{u}] = T\big[S[\boldsymbol{u}]\big]$. The product $T\,S$ is defined only when the target of S lies in the source of T. One verifies that this binary operation has most of the properties expected of a product.

(associativity)	$(T_1\,T_2)\,T_3 = T_1\,(T_2\,T_3)$	whenever the products are all defined.
(identity)	$T\,\mathrm{id}_V = T = \mathrm{id}_W\,T$	when $T \colon V \to W$.
(linearity)	$T(c_1\,S_1 + c_2\,S_2) = c_1(T\,S_1) + c_2(T\,S_2)$	when $S_1, S_2 \colon U \to V$, $T \colon V \to W$, and $c_1, c_2 \in \mathbb{K}$.
	$(c_1\,T_1 + c_2\,T_2)S = c_1\,(T_1\,S) + c_2\,(T_2\,S)$	when $S \colon U \to V$, $T_1, T_2 \colon V \to W$, and $c_1, c_2 \in \mathbb{K}$.

The product to two linear maps is not typically commutative.

3.2.1 Problem. Let $D \colon \mathbb{K}[t] \to \mathbb{K}[t]$ denote differentiation and let $M \colon \mathbb{K}[t] \to \mathbb{K}[t]$ denote multiplication by t^2. Show that $D\,M \neq M\,D$.

Solution. For all nonzero polynomials f in $\mathbb{K}[t]$, it follows that $(M\,D)[f] = t^2\,f'$ whereas $(D\,M)[f] = D[t^2\,f] = t^2\,f' + 2t\,f \neq t^2\,f'$. □

The definition of an invertible linear map generalizes the definition of an invertible matrix.

3.2.2 Definition. A linear map $T \colon V \to W$ is ***invertible*** if there exists a linear map $S \colon W \to V$ such that $S\,T = \mathrm{id}_V$ and $T\,S = \mathrm{id}_W$. In this case, the map S is an ***inverse*** of T.

The identity map $\mathrm{id}_V \colon V \to V$ is the map whose output is equal to its input.

3.2.3 Problem. Let $V := \mathbb{R}^{\mathbb{R}}$ be the $\mathbb{R}$-vector space of real-valued functions on the real line. Fix $a \in \mathbb{R}$ and consider the two linear maps $T, S \colon V \to V$ defined by $T[f(x)] = f(x+a)$ and $S[f(x)] = f(x-a)$ respectively. Show that S is an inverse of T.

The operators T and S translate the graph of a function horizontally by a in opposite directions.

Solution. Since $(S\,T)[f(x)] = S[f(x+a)] = f\big((x+a)-a\big) = f(x)$ and $(T\,S)[f(x)] = T[f(x-a)] = f\big((x-a)+a\big) = f(x)$, we see that $S\,T = \mathrm{id}_V = T\,S$ and these translations maps are mutual inverses. □

3.2.4 Proposition (Uniqueness of the inverse)**.** *For any invertible linear map $T \colon V \to W$, the inverse map is unique and denoted by $T^{-1} \colon W \to V$.*

Since $T^{-1}\,T = \mathrm{id}_V$ and $T\,T^{-1} = \mathrm{id}_W$, the uniqueness of the inverse implies that $(T^{-1})^{-1} = T$.

Proof. Suppose that the linear maps $S_1 \colon W \to V$ and $S_2 \colon W \to V$ are both inverses of the linear map $T \colon V \to W$. It follows that

$$S_1 = S_1 \ \mathrm{id}_W = S_1\,(T\,S_2) = (S_1\,T)\,S_2 = \mathrm{id}_V \ S_2 = S_2\,. \qquad \square$$

3.2.5 Proposition (Characterization of invertibility)**.** *A linear map is invertible if and only if it is bijective.*

The inverse of a linear map is automatically a linear map.

Proof. Consider a linear map $T \colon V \to W$.

$\Rightarrow$: Suppose that T is invertible. For any two vectors $\boldsymbol{v}$ and $\boldsymbol{v}'$ in V satisfying $T[\boldsymbol{v}] = T[\boldsymbol{v}']$, we have $\boldsymbol{v} = T^{-1}\big[T[\boldsymbol{v}]\big] = T^{-1}\big[T[\boldsymbol{v}']\big] = \boldsymbol{v}'$, so the map T is injective. For any vector $\boldsymbol{w}$ in W, we also have $\boldsymbol{w} = T\big[T^{-1}[\boldsymbol{w}]\big]$, so the map T is surjective.

$\Leftarrow$: Suppose that T is bijective. The surjectivity and injectivity of T imply that, for each vector $\boldsymbol{w}$ in W, there exists a unique vector $S[\boldsymbol{w}]$ in V such that $T\big[S[\boldsymbol{w}]\big] = \boldsymbol{w}$. In other words, there exists a unique set map $S \colon W \to V$ for which $T\,S = \mathrm{id}_W$. For any vector $\boldsymbol{v}$ in V, it follows that

$$T\big[(S\,T)[\boldsymbol{v}]\big] = T\big[S\big[T[\boldsymbol{v}]\big]\big] = (T\,S)\big[T[\boldsymbol{v}]\big] = \mathrm{id}_W\big[T[\boldsymbol{v}]\big] = T[\boldsymbol{v}]\,.$$

Since the map T is injective, we deduce that $(S\,T)[\boldsymbol{v}] = \boldsymbol{v}$ for all $\boldsymbol{v}$ in V and $S\,T = \mathrm{id}_V$. It remains to show that S is linear. For all vectors $\boldsymbol{w}$ and $\boldsymbol{w}'$ in W and all scalars b and c in $\mathbb{K}$, the linearity of the map T gives

$$T\big[b(S[\boldsymbol{w}]) + c(S[\boldsymbol{w}'])\big] = b\big(T\big[S[\boldsymbol{w}]\big]\big) + c\big(T\big[S[\boldsymbol{w}']\big]\big) = b\,\boldsymbol{w} + c\,\boldsymbol{w}'\,.$$

Hence, $b(S[\boldsymbol{w}]) + c(S[\boldsymbol{w}'])$ is the unique vector in V that the map T sends to $b\,\boldsymbol{w} + c\,\boldsymbol{w}'$. Therefore, the definition of the map S implies that $S[b\,\boldsymbol{w} + c\,\boldsymbol{w}'] = b(S[\boldsymbol{w}]) + c(S[\boldsymbol{w}'])$. □

3.2.6 Definition. Two $\mathbb{K}$-vector spaces V and W are ***isomorphic***, denoted $V \cong W$, if there is an invertible linear map from V to W.

One may regard an invertible linear map as a relabeling/renaming of the elements in a vector space. Thus, two isomorphic vectors spaces have the same properties (from the perspective of linear algebra).

3.2.7 Theorem. *Let V and W be finite-dimensional $\mathbb{K}$-vector spaces. We have* $\dim(V) = \dim(W)$ *if and only if V is isomorphic to W.*

Proof.

$\Rightarrow$: Set $n := \dim(V) = \dim(W)$. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$ and $\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n$ be bases for V and W respectively. A linear map is determined by its values on a basis [3.0.7], so consider $T \colon V \to W$ defined, for all $1 \leqslant j \leqslant n$, by $T[\boldsymbol{v}_j] = \boldsymbol{w}_j$. For any vector $\boldsymbol{w}$ in W, there exists scalars $a_1, a_2, \dots, a_n$ in $\mathbb{K}$ such that $\boldsymbol{w} = a_1\, \boldsymbol{w}_1 + a_2\, \boldsymbol{w}_2 + \cdots + a_n\, \boldsymbol{w}_n$, because the vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n$ span W. It follows that

$$\begin{aligned} T[a_1\, \boldsymbol{v}_1 + a_2\, \boldsymbol{v}_2 + \cdots + a_n\, \boldsymbol{v}_n] &= a_1\, T[\boldsymbol{v}_1] + a_2\, T[\boldsymbol{v}_2] + \cdots + a_n\, T[\boldsymbol{v}_n] \\ &= a_1\, \boldsymbol{w}_1 + a_2\, \boldsymbol{w}_2 + \cdots + a_n\, \boldsymbol{w}_n = \boldsymbol{w}\,, \end{aligned}$$

which shows that the linear map T is surjective. Similarly, for any vector $\boldsymbol{v}$ in $\operatorname{Ker}(T)$, there exists scalars $b_1, b_2, \dots, b_n$ in $\mathbb{K}$ such that $\boldsymbol{v} = b_1\, \boldsymbol{v}_1 + b_2\, \boldsymbol{v}_2 + \cdots + b_n\, \boldsymbol{v}_n$ because the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$ span V. It follows that

$$\begin{aligned} \boldsymbol{0} = T[\boldsymbol{v}] &= T[b_1\, \boldsymbol{v}_1 + b_2\, \boldsymbol{v}_2 + \cdots + b_n\, \boldsymbol{v}_n] \\ &= b_1\, T[\boldsymbol{v}_1] + b_2\, \boldsymbol{T}[\boldsymbol{v}_2] + \cdots + b_n\, T[\boldsymbol{v}_n] \\ &= b_1\, \boldsymbol{w}_1 + b_2\, \boldsymbol{w}_2 + \cdots + b_n\, \boldsymbol{w}_n\,. \end{aligned}$$

Since the vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n$ are linearly independent, we deduce that $b_1 = b_2 = \cdots = b_n = 0$, $\boldsymbol{v} = \boldsymbol{0}$, and $\operatorname{Ker}(T) = \{\boldsymbol{0}\}$. The linear characterization of injectivity [3.1.4] implies that the map T is injective and the characterization of invertibility [3.2.5] establishes that T is invertible. It follows that $V \cong W$.

$\Leftarrow$: Suppose that there is an invertible linear map $T \colon V \to W$. The characterization of invertibility [3.2.5] implies that T is bijective and the characterizations of injectivity and surjectivity [3.1.4] imply that $\operatorname{Ker}(T) = \{\boldsymbol{0}\}$ and $\operatorname{Im}(T) = W$. Thus, the dimension formula [3.1.4] gives

$$\dim(V) = \dim\big(\operatorname{Ker}(T)\big) + \dim\big(\operatorname{Im}(T)\big) = 0 + \dim(W) = \dim(W)\,. \quad \square$$

3.2.8 Remark. Theorem3.2.7 establishes that, for any finite-dimensional $\mathbb{K}$-vector space V where $n := \dim V$, we have $V \cong \mathbb{K}^n$.

Exercises

3.2.9 Problem. Determine which of the following statements are true. If a statement is false, then provide a counterexample.

i. The product of nonzero linear transformations is never zero.
ii. The product of two linear transformations is never commutative.
iii. Consider any two linear transformations S and T, If we have $S\,T = I$, then we must have $T\,S = I$.
iv. The $\mathbb{K}$-vector spaces $\mathbb{K}[t]_{\leqslant n}$ and $\mathbb{K}^{n+1}$ are isomorphic.

3.2.10 Problem. Fix a nonnegative integer n. Show that a polynomial f in $\mathbb{R}[t]_{\leqslant n}$ is uniquely determined by the vector $\begin{bmatrix} x_0 & x_1 & \cdots & x_n \end{bmatrix}^{\mathsf{T}}$ in $\mathbb{R}^{n+1}$ where $x_k := \int_0^1 t^k f(t)\, dt$.

3.3 Invertible Operators

ARE INVERTIBLE MAPS FROM A VECTOR SPACE TO ITSELF SPECIAL? Some of the deepest and most important parts of linear algebra deal with linear maps from a vector space to itself.

3.3.0 Definition. A linear map from a vector space to itself is called a ***linear operator*** or ***endomorphism***.

The set of all linear operators on the $\mathbb{K}$-vector space V is sometimes denoted by $\operatorname{End}(V) := \operatorname{Hom}(V, V)$.

3.3.1 Remark. For any $\mathbb{K}$-vector space V, the simplest linear operators are the identity map $\mathrm{id}_V \colon V \to V$ is a linear operator and its scalar multiples. For any scalar $c \in \mathbb{K}$, the linear map $c\,\mathrm{id}_V \colon V \to V$ is defined by $c\,\mathrm{id}_V[\boldsymbol{v}] := c\,\boldsymbol{v}$ for all vectors $\boldsymbol{v}$ in V.

Square matrices correspond to linear operators. More precisely, for any nonnegative integer n, left multiplication by an $(n \times n)$-matrix defines a linear operator on the coordinate space $\mathbb{K}^n$.

3.3.2 Problem. Let V be a finite-dimensional $\mathbb{K}$-vector space. Prove that the linear map $T \colon V \to V$ is a scalar multiple of the identity map if and only if, for any linear map $S \colon V \to V$, we have $S\,T = T\,S$.

Solution.

$\Rightarrow$: Suppose that we have $T = c\,\mathrm{id}_V$ for some scalar c in $\mathbb{K}$. It follows that $S\,T = S\,(c\,\mathrm{id}_V) = c\,(S\,\mathrm{id}_V) = c\,S = (c\,\mathrm{id}_V)\,S = T\,S$.

$\Leftarrow$: Suppose that the map $T \colon V \to V$ commutes with every linear operator on the $\mathbb{K}$-vector space V. Choose a basis $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$ for the finite-dimensional vector space V. For all $1 \leqslant k \leqslant n$, the image $T[\boldsymbol{v}_k]$ is a unique linear combination of the basis vectors. Hence, there exists unique scalars $a_{1,k}, a_{2,k}, \dots, a_{n,k}$ in $\mathbb{K}$ such that $T[\boldsymbol{v}_k] = a_{1,k}\,\boldsymbol{v}_1 + a_{2,k}\,\boldsymbol{v}_2 + \cdots + a_{n,k}\,\boldsymbol{v}_n$.

A linear map is determined by its values on a basis [3.0.7]. For all $1 \leqslant j \leqslant n$, consider the linear map $P_j \colon V \to V$ defined by $P_j[\boldsymbol{v}_j] = \boldsymbol{v}_j$ and $P_j[\boldsymbol{v}_k] = \boldsymbol{0}$ if $k \neq j$. When $k \neq j$, we obtain

$$\boldsymbol{0} = T[\boldsymbol{0}] = (T\,P_j)[\boldsymbol{v}_k] = (P_j\,T)[\boldsymbol{v}_k] = P_j[a_{1,k}\,\boldsymbol{v}_1 + a_{2,k}\,\boldsymbol{v}_2 + \cdots + a_{n,k}\,\boldsymbol{v}_n] = a_{j,k}\,\boldsymbol{v}_j\,,$$

so $a_{j,k} = 0$ and $T[\boldsymbol{v}_k] = a_{k,k}\,\boldsymbol{v}_k$.

Next, consider the linear map $S \colon V \to V$ defined, for all $1 \leqslant k \leqslant n-1$, by $S[\boldsymbol{v}_k] = \boldsymbol{v}_{k+1}$ and $S[\boldsymbol{v}_n] = \boldsymbol{v}_1$. It follows that

$$\begin{aligned} a_{k+1,k+1}\,\boldsymbol{v}_{k+1} = T[\boldsymbol{v}_{k+1}] = (T\,S)[\boldsymbol{v}_k] = (S\,T)[\boldsymbol{v}_k] = S[a_{k,k}\,\boldsymbol{v}_k] = a_{k,k}\,\boldsymbol{v}_{k+1} \\ \Rightarrow \quad (a_{k+1,k+1} - a_{k,k})\,\boldsymbol{v}_{k+1} = \boldsymbol{0}\,, \end{aligned}$$

so we deduce that $c := a_{1,1} = a_{2,2} = \cdots = a_{n,n}$. We conclude that $T[\boldsymbol{v}_k] = c\,\boldsymbol{v}_k$. for all $1 \leqslant k \leqslant n$, proving that $T = c\,\mathrm{id}_V$. □

3.3.3 Problem. Demonstrate that the linear operator on $\mathbb{K}[t]$ defined via multiplication by t^2 is injective, but is not surjective.

Solution. Let $M \colon \mathbb{K}[t] \to \mathbb{K}[t]$ be the map defined, for any polynomial f in $\mathbb{K}[t]$, by $M[f] := t^2\,f$. The equation $0 = M[f] = t^2\,f$ implies that $f = 0$. Since $\operatorname{Ker}(M) = \{0\}$, the characterization of injectivity [3.1.4] shows that M is injective. Since every nonzero polynomial in the image of M must have degree at least 2, the map M is not surjective: neither 1 nor t belong to $\operatorname{Im}(M)$. □

3.3.4 Problem. The backward shift operator $B \colon \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}$ is defined by $B[(a_0, a_1, a_2, \dots)] := (a_1, a_2, a_3, \dots)$. Show that B is surjective, but is not injective.

Solution. Since $\operatorname{Ker}(B) = \{(a_0, 0, 0, \dots) \in \mathbb{K}^{\mathbb{N}} \mid a_0 \in \mathbb{K}\} \neq \{0\}$, the map B is not injective. For any sequence $(a_0, a_1, a_2, \dots)$ in $\mathbb{K}^{\mathbb{N}}$, we have $B[(0, a_0, a_1, \dots)] = (a_0, a_1, a_2, \dots)$, so the map B is surjective. □

In view of the last two problems, the next theorem is remarkable.

3.3.5 Theorem (Characterization of invertible operators)**.** *Let V be a finite-dimensional vector space. For any linear map $T \colon V \to V$, the following are equivalent.*

a. *The linear map T is invertible.*
b. *The linear map T is injective.*
c. *The linear map T is surjective.*

Proof.

$a \Rightarrow b$: The characterization of invertibility [3.2.5] shows that every invertible linear map is bijective and, in particular, injective.

$b \Rightarrow c$: Since T is injective, the characterization of injectivity [3.1.4] implies that $\dim(\operatorname{Ker}(T)) = 0$. Hence, the dimension formula [3.1.6] gives $\dim(V) = \dim(\operatorname{Ker}(T)) + \dim(\operatorname{Im}(T)) = \dim(\operatorname{Im}(T))$. As $\operatorname{Im}(T) \subseteq V$, we see that $V = \operatorname{Im}(T)$ and T is surjective.

$c \Rightarrow a$: The surjectivity of T means $\operatorname{Im}(T) = V$. Hence, the dimension formula [3.1.6] implies that $\dim(\operatorname{Ker}(T)) = 0$ and the linear characterization of injectivity [3.1.4] implies that the map T is injective. Thus, the characterization of invertibility [3.2.5] demonstrates that the linear map T is invertible. □

3.3.6 Problem. Let f be a polynomial in $\mathbb{R}[t]$. Establish that there exists a polynomial g in $\mathbb{R}[t]$ such that $\frac{d^2}{dt^2}\big((t+1)^2 g\big) = f$.

Solution. Let n denote the degree of the polynomial f. Consider the map $T \colon \mathbb{R}[t]_{\leqslant n} \to \mathbb{R}[t]_{\leqslant n}$ defined, for all polynomials g in $\mathbb{R}[t]_{\leqslant n}$, by $T[g] := \frac{d^2}{dt^2}\big((t+1)^2 g\big)$. Since multiplying by a nonzero polynomial by $(t+1)^2$ increases the degree by 2 and differentiating twice decreases the degree by 2, we see that T is a linear operator on $\mathbb{R}[t]_{\leqslant n}$.

Every polynomial whose second derivative equals 0 has the form $a_0 + a_1 t$ for some $a_0, a_1 \in \mathbb{R}$. We deduce that $\operatorname{Ker}(T) = \{0\}$ and the characterization of injectivity [3.1.4] implies that T is injective. Hence, the characterization of invertible operators [3.3.5] shows that T is surjective. Therefore, there exists g in $\mathbb{R}[T]_{\leqslant n}$ such that $T[g] = f$. □

Exercises

3.3.7 Problem. Determine which of the following statements are true. If a statement is false, then provide a counterexample.

i. The zero homomorphism is always a linear operator.

ii. The identity map is always a linear operator.

iii. Consider any two linear operators S and T on a finite-dimensional vector space. If we have $S\,T = I$, then we must have $T\,S = I$.

3.3.8 Problem. Let V be a finite-dimensional vector space. Consider two linear operators S and T on V.

i. Show that the product $S\,T$ is invertible if and only if both S and T are invertible.

ii. Prove that $S\,T = I$ if and only if $T\,S = I$.

iii. Give an example illustrating that both (a) and (b) are false over an infinite-dimensional vector space.