Problems 07

Due: Friday, 29 October 2021 before 17:00 EDT

1. Find vectors that span the kernel of the matrix

$$\mathbf{M} := \begin{bmatrix} 0 & 0 & 1 & \sqrt{2} & 0 & 0 & -\mathrm{i} & 0 & e & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \pi & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

2. A matrix that has the same number of rows as columns is ***square***. A square matrix $\mathbf{A}$ is ***symmetric*** if $\mathbf{A}^{\mathsf{T}} = \mathbf{A}$. Similarly, a square matrix $\mathbf{A}$ is ***skew-symmetric*** if $\mathbf{A}^{\mathsf{T}} = -\mathbf{A}$.

(i) For any square matrix $\mathbf{A}$, show that the matrix $\frac{1}{2}(\mathbf{A} + \mathbf{A}^{\mathsf{T}})$ is symmetric.

(ii) Prove that any square matrix can be written uniquely as the sum of a symmetric matrix and a skew-symmetric matrix.

(iii) Illustrate part (ii) for the the matrix $\begin{bmatrix} 6 & 5 & -3 \\ -3 & 4 & -4 \\ -7 & 2 & 2 \end{bmatrix}$.

3. The ***Gell-Mann*** matrices are the following eight complex (3×3)-matrices:

$$\mathbf{G}_1 := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{G}_2 := \begin{bmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{G}_3 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{G}_4 := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\mathbf{G}_5 := \begin{bmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{bmatrix}, \quad \mathbf{G}_6 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{G}_7 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{bmatrix}, \quad \mathbf{G}_8 := \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Describe all complex (3×3)-matrices $\mathbf{B}$ for which there exists scalars $x_1, x_2, \ldots, x_8 \in \mathbb{C}$ such that

$$x_1 \mathbf{G}_1 + x_2 \mathbf{G}_2 + \cdots + x_8 \mathbf{G}_8 = \mathbf{B}.$$