Problems 08

Due: Friday, 5 November 2021 before 17:00 EDT

1. Fix positive integers m and n. Let $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n \in \mathbb{K}^m$ be linearly independent vectors.

(i) Show that the vectors $\vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_2 - \vec{\mathbf{v}}_3, \dots, \vec{\mathbf{v}}_{n-1} - \vec{\mathbf{v}}_n, \vec{\mathbf{v}}_n \in \mathbb{K}^m$ are also linearly independent.

(ii) Suppose that, for some $\vec{\mathbf{w}} \in \mathbb{K}^m$, the vectors $\vec{\mathbf{v}}_1 + \vec{\mathbf{w}}, \vec{\mathbf{v}}_2 + \vec{\mathbf{w}}, \dots, \vec{\mathbf{v}}_n + \vec{\mathbf{w}}$ are linearly dependent. Prove that $\vec{\mathbf{w}} \in \operatorname{Span}(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n)$.

2. Let n be a positive integer. The ***trace*** of an $(n \times n)$-matrix $\mathbf{A} := [a_{j,k}]$ is the sum of its diagonal entries:

$$\operatorname{tr}(\mathbf{A}) := a_{1,1} + a_{2,2} + \cdots + a_{n,n} = \sum_{j=1}^{n} a_{j,j}\,.$$

(i) Prove that the trace is linear. In other words, show that, for any $(n \times n)$-matrices $\mathbf{A}$, $\mathbf{B}$ and any scalars $c, d \in \mathbb{K}$, we have $\operatorname{tr}(c\,\mathbf{A} + d\,\mathbf{B}) = c\operatorname{tr}(\mathbf{A}) + d\operatorname{tr}(\mathbf{B})$.

(ii) For any two $(n \times n)$-matrices $\mathbf{A}$ and $\mathbf{B}$, prove that $\operatorname{tr}(\mathbf{A}\,\mathbf{B}) = \operatorname{tr}(\mathbf{B}\,\mathbf{A})$.

(iii) Show that matrix equation $\mathbf{X}\,\mathbf{Y} - \mathbf{Y}\,\mathbf{X} = \mathbf{I}$ has no solutions for $(n \times n)$-matrices $\mathbf{X}$ and $\mathbf{Y}$.

3. (i) Let $\mathbf{R} := \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$. For all nonnegative integers k, show that $\mathbf{R}^k = \begin{bmatrix} \cos(k\theta) & -\sin(k\theta) \\ \sin(k\theta) & \cos(k\theta) \end{bmatrix}$.

(ii) In 1969, Volker Strassen surprised the mathematical community by showing that two (2×2)-matrix can be multiplied using only seven multiplications of scalars. Establish his method by showing that, when

$$\mathbf{A} := \begin{bmatrix} a & c \\ b & d \end{bmatrix} \quad p_1 := (a+d)(w+z) \quad p_3 := a(y-z) \quad p_5 := (a+c)z \quad p_7 := (c-d)(x+z)\,,$$
$$\mathbf{B} := \begin{bmatrix} w & y \\ x & z \end{bmatrix} \quad p_2 := (b+d)w \quad p_4 := d(x-w) \quad p_6 := (b-a)(w+y)$$

we have $\mathbf{A}\,\mathbf{B} = \begin{bmatrix} p_1 + p_4 - p_5 + p_7 & p_3 + p_5 \\ p_2 + p_4 & p_1 + p_3 - p_2 + p_6 \end{bmatrix}$.