Problems 17

Due: Friday, 11 February 2022 before 17:00 EST

P17.1. Let U, V, W be three $\mathbb{K}$-vector spaces such that both U and V have finite dimension. Consider the linear maps $S\colon U \to V$ and $T\colon V \to W$.

(i) Demonstrate that $\dim\big(\operatorname{Ker}(TS)\big) \leqslant \dim\big(\operatorname{Ker}(S)\big) + \dim\big(\operatorname{Ker}(T)\big)$.

(ii) Demonstrate that $\dim\big(\operatorname{Im}(TS)\big) \leqslant \min\big\{\dim\big(\operatorname{Im}(S)\big), \dim\big(\operatorname{Im}(T)\big)\big\}$.

P17.2. Let V be a finite-dimensional vector space. Consider two linear operators $T\colon V \to V$ and $S\colon V \to V$.

(i) Show that the product ST is invertible if and only if both S and T are invertible.

(ii) Prove that $ST = \mathrm{id}_V$ if and only if $TS = \mathrm{id}_V$.

(iii) Give an example showing that parts (i)–(ii) are false over an infinite-dimensional vector space.

P17.3. Let n be a positive integer and let T_n denote the $\mathbb{R}$-vector space of trigonometric polynomials having the functions $\big(1, \cos(x), \sin(x), \dots, \cos(nx), \sin(nx)\big)$ is an ordered basis. For a fixed nonnegative real number a, consider the linear map $D\colon \mathrm{T}_n \to \mathrm{T}_n$ defined, for all f in T_n, by $D[f] = f'' + a^2 f$. For which scalars a is the linear operator D invertible?