Problems 10

Due: Friday, 24 March 2023 before 17:00 EST

P10.1. Consider the subrings $\mathbb{Z}[\sqrt{5}] := \{a + b\sqrt{5} \mid a, b \in \mathbb{Z}\}$ and

$$\mathbb{Z}\left[\tfrac{1+\sqrt{5}}{2}\right] := \left\{a + b\left(\tfrac{1+\sqrt{5}}{2}\right) \,\middle|\, a, b \in \mathbb{Z}\right\}$$

of the field $\mathbb{R}$ of real numbers. For each subring, describe the elements in the field of fractions. Are these two fields the same? Is one contained in the other?

P10.2. Let $\omega := \frac{1}{2}(-1 + \sqrt{3}\,\mathrm{i}) \in \mathbb{C}$ be one of the complex roots of the polynomial $x^2 + x + 1$ in $\mathbb{C}[x]$. Prove that the commutative domain $\mathbb{Z}[\omega] := \{a + b\,\omega \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}$ is a Euclidean domain with the function $\nu \colon \mathbb{Z}[\omega] \to \mathbb{N}$ defined by $\nu(a + b\,\omega) = a^2 - a\,b + b^2$.

P10.3. Let $\mathbb{F}_2 := \mathbb{Z}/\langle 2\rangle$ be the field with two elements. Find a polynomial f in $\mathbb{F}_2[x]$ such that

$$\begin{aligned}
f + \langle x\rangle &= 1 + \langle x\rangle && \text{in } \mathbb{F}_2[x]/\langle x\rangle,\\
f + \langle x^2 + x + 1\rangle &= (x + 1) + \langle x^2 + x + 1\rangle && \text{in } \mathbb{F}_2[x]/\langle x^2 + x + 1\rangle,\\
f + \langle x^4 + x^3 + 1\rangle &= (x^3 + x + 1) + \langle x^4 + x^3 + 1\rangle && \text{in } \mathbb{F}_2[x]/\langle x^4 + x^3 + 1\rangle.
\end{aligned}$$