Problems 12

Due: Monday, 10 April 2023 before 17:00 EDT

P12.1. Euclid proves that there are infinitely many prime integers in the following way: if $p_1, p_2, \dots, p_k$ are positive prime integers, then any prime factor of $1 + p_1\, p_2 \cdots p_k$ must be different from p_j for any $1 \leqslant j \leqslant k$.

(i) Adapt this argument to show that the set of prime integers of the form $4\,n - 1$ is infinite.

(ii) Adapt this argument to show that, for any field $\mathbb{K}$, there are infinitely many monic irreducible polynomials in $\mathbb{K}[x]$.

P12.2. (i) Let $f := a_3\, x^3 + a_2\, x^2 + a_1\, x + a_0$ be a polynomial in $\mathbb{Z}[x]$ having degree 3. Assume that a_0, $a_1 + a_2$, and a_3 are all odd. Prove that f is irreducible in $\mathbb{Q}[x]$.

(ii) Prove that the polynomial $g := x^5 + 6\,x^4 - 12\,x^3 + 9\,x^2 - 3\,x + k$ in $\mathbb{Q}[x]$ is irreducible for infinitely many integers k.

(iii) Prove that $h := x^5 + x^4 + x - 1$ is irreducible in $\mathbb{Q}[x]$ using the Eisenstein criterion.

P12.3. *Existence of Partial Fraction Decompositions.* Let R be a principal ideal domain and let K be its field of fractions.

(i) Suppose $R = \mathbb{Z}$. Write $r = \frac{7}{24} \in \mathbb{Q}$ in the form $r = \frac{b}{3} + \frac{a}{8}$ for some integers a and b.

(ii) Let $g := p\,q \in R$ where p and q are coprime. Prove that every fraction $f/g \in K$ can written in the form

$$\frac{f}{g} = \frac{u}{q} + \frac{v}{p}$$

for some elements u and v in R.

(iii) Let $g := p_1^{e_1}\, p_2^{e_2} \cdots p_m^{e_m} \in R$ be the factorization of g into irreducible elements p_j, for all $1 \leqslant j \leqslant m$, such that the relation $p_j = u\,p_k$ for some unit $u \in R$ implies that $j = k$. Prove that every fraction $f/g \in K$ can be written in the form

$$\frac{f}{g} = \sum_{j=1}^{k} \frac{h_j}{p_j^{e_j}}$$

for some elements $h_1, h_2, \dots, h_m$ in R.