Problem Set #7
Due: Friday, March 2, 2007

1. Let $n = 3$. Define an inner product on $\mathbb{R}[x]_{\leq n}$ by

$$\langle f, g \rangle := \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}} \, dx \, .$$

(a) Apply the Gram-Schmidt procedure to the basis $(1, x, \ldots, x^n)$ to produce an orthonormal basis $\big(e_0(x), \ldots, e_n(x)\big)$ of $\mathbb{R}[t]_{\leq n}$.

(b) For $0 \leq j \leq n$ consider the operator $D_j \in \operatorname{End}\big(\mathbb{R}[t]_{\leq n}\big)$ defined by

$$D_j(f) = (1-x^2)f''(x) - xf'(x) + j^2 f(x) \, .$$

Show that $\operatorname{Span}\big(e_j(x)\big) = \operatorname{Ker}(D_j)$.

Hint. For $1 \leq j \leq 6$, use a computer algebra system to compute the integrals

$$I_j := \int_{-1}^{1} \frac{x^j}{\sqrt{1-x^2}} \, dx \, .$$

2. Let $T \in \operatorname{End}(V)$ satisfy $T^2 = T$.

(a) Prove that $V = \operatorname{Ker}(T) \oplus \operatorname{Im}(T)$.

(b) Suppose that $\|Tv\| \leq \|v\|$ for all $v \in V$. Prove that T is an orthogonal projection.

Hint. Use Problem #6.2 and part (a) to establish part (b).

3. Give $\mathbb{R}^4$ the inner product

$$\langle (x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \rangle = 2x_1y_1 + 2x_2y_2 + x_3y_3 + x_4y_4 \, .$$

Let $U = \operatorname{Span}\big((1, 0, 1, 1), (3, 2, -1, -1)\big)$ and let P_U be the orthogonal projection onto U. Find a matrix A satisfying

$$P_U(x_1, x_2, x_3, x_4) = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad \text{for all } (x_1, x_2, x_3, x_4) \in \mathbb{R}^4.$$