Problem Set #3

Due: Friday, 14 January 2020

1. Fix $n \in \mathbb{N}$. For all $2 \leqslant k \leqslant n$, the ***Jucys–Murphy element*** X_k in $\mathbb{C}[\mathfrak{S}_n]$ is defined to be

$$X_k := (k\ \ 1) + (k\ \ 2) + \cdots + (k\ \ k-1) = \sum_{\ell=1}^{k-1} (k\ \ \ell)\,.$$

(a) For all $i < j < k$ and all $\ell < k$, compute the product $(j\ \ i)(k\ \ \ell)(j\ \ i)$.

(b) Prove that X_n commutes with every element in $\mathbb{C}[\mathfrak{S}_{n-1}]$ (regarded as a subalgebra of $\mathbb{C}[\mathfrak{S}_n]$).

(c) For all $2 \leqslant j < k \leqslant n$, show that $X_j X_k = X_k X_j$.

2. For two integer partitions λ and μ, consider the following four binary operations.

- The ***sum*** $\lambda + \mu$ is the entrywise sum; $(\lambda+\mu)_j := \lambda_j + \mu_j$ for all $j \geqslant 1$.
- The ***amalgam*** $\lambda \sqcup \mu$ is the partition whose parts are those of λ and μ arranged in descending order.
- The ***product*** $\lambda\,\mu$ is the entrywise product; $(\lambda\,\mu)_j := \lambda_j\,\mu_j$ for all $j \geqslant 1$.
- The ***coproduct*** $\lambda \oplus \mu$ is the partition whose parts are $\min(\lambda_j, \mu_k)$ for all j at most the length of λ and all k at most the length of μ.

(a) For $\lambda = (3,2,2)$ and $\mu = (3,2,1,1)$, compute the following:

$\lambda + \mu$	$\lambda \sqcup \mu$	$\lambda\,\mu$	$\lambda \oplus \mu$
$(\lambda+\mu)'$	$(\lambda \sqcup \mu)'$	$(\lambda\,\mu)'$	$(\lambda \oplus \mu)'$
$\lambda' + \mu'$	$\lambda' \sqcup \mu'$	$\lambda'\,\mu'$	$\lambda' \oplus \mu'$
$(\lambda'+\mu')'$	$(\lambda' \sqcup \mu')'$	$(\lambda'\,\mu')'$	$(\lambda' \oplus \mu')'$

(b) Prove that $(\lambda \sqcup \mu)' = \lambda' + \mu'$ and $(\lambda \oplus \mu)' = \lambda'\,\mu'$.

3. Fix $n \in \mathbb{N}$. Let $\mathcal{P} \subset \mathbb{N}^{n+1}$ be the set of all tuples $\boldsymbol{v} := (v_0, v_1, \dots, v_n)$ satisfying the following properties:

(nondecreasing) for all $0 \leqslant i < n$, we have $v_i \leqslant v_{i+1}$;
(concave) for all $0 < i < n$, we have $v_{i-1} + v_{i+1} \leqslant 2v_i$;
(boundary values) we have $v_0 = 0$ and $v_n = n$.

Consider the map Σ from the set of all partitions of n to $\mathbb{N}^{n+1}$ defined by

$$\Sigma(\lambda) := (0, \lambda_1, \lambda_1 + \lambda_2, \dots, \lambda_1 + \lambda_2 + \cdots + \lambda_n)\,,$$

and the map $\Delta\colon \mathcal{P} \to \mathbb{Z}^n$ defined by $\Delta(\boldsymbol{v}) := (v_1 - v_0, v_2 - v_1, \dots, v_n - v_{n-1})$.

(a) Check that the maps Σ and Δ are bijections between the set of partitions of n and the set $\mathcal{P}$.

(b) Under these bijections, show that the dominance order on the partitions of n corresponds to the componentwise order on $\mathcal{P}$.

4. For all $n \geqslant 2$, decompose the defining permutation representation of the symmetric group $\mathfrak{S}_n$ into irreducible representations.

Hint. Show that standard representation is irreducible by proving that any nonempty submodule must be the entire standard representation.

5. Decompose the regular representation of $\mathfrak{S}_3$ into irreducible representations.