Problem Set #2

Due: Thursday, 20 September 2012

Students registered in MATH 401 should submit solutions to three of the following problems. Students in MATH 801 should submit solutions to all five.

1. For a positive integer n, the ***boolean lattice*** BL_n is the graph whose vertex set is the set of all subsets of $\{1, 2, \ldots, n\}$ where two subsets are adjacent if their symmetric difference has precisely one element. Show that the n-cube Q_n and the boolean lattice BL_n are isomorphic.

2. For positive integers m and n, the ***Kneser graph*** $KG_{n,m}$ has vertex set consisting of the m-subset of $\{1, \ldots, n\}$ and two vertices are adjacent if the corresponding subsets are disjoint.

(**a**) Show that $KG_{n,1} \cong K_n$.

(**b**) Show that $KG_{n,2}$ is isomorphic to the complement of line graph $L(K_n)$.

3. The Desargues set system $(V, \mathcal{F})$ has $V := \{0, 1, \ldots, 9\}$ and

$$\mathcal{F} := \left\{ \begin{array}{ccccc} \{0,4,7\}, & \{0,5,8\}, & \{0,6,9\}, & \{1,2,3\}, & \{1,4,5\} \\ \{1,7,8\}, & \{2,4,6\}, & \{2,7,9\}, & \{3,5,6\}, & \{3,8,9\} \end{array} \right\} .$$

Show that the following three graphs are isomorphic:

- the intersection graph of the Desargues set system,
- the line graph of K_5,
- the complement of the Petersen graph.

4. (**a**) Let G be a graph satisfying $e(G) > \binom{v(G)-1}{2}$. Prove that G is connected.

(**b**) For any positive integer n, find a disconnected graph G such that $v(G) = n$ and $e(G) = \binom{n-1}{2}$.

5. (**a**) Let G be a graph with minimum degree $\delta(G)$. If $\delta(G) > \frac{1}{2}\big(v(G) - 2\big)$, then show that G is connected.

(**b**) For any positive even number n, find a disconnected $\frac{1}{2}(n-2)$-regular graph G such that $v(G) = n$.

Hint. It suffices to show that every vertex not adjacent to one with minimal degree has a common neighbour.