Problem Set #3

Due: Thursday, 27 September 2012

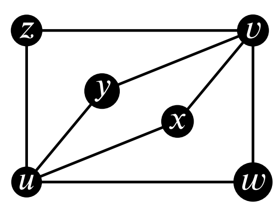
Students registered in MATH 401 should submit solutions to three of the following problems. Students in MATH 801 should submit solutions to all five.

1. Let G be a graph with at least two vertices, and let $d(G) := \frac{1}{v(G)} \sum_{v \in V(G)} d(v)$ be the average degree of G. Prove or disprove:
(**a**) Deleting a vertex of maximum degree $\Delta(G)$ cannot increase $d(G)$.
(**b**) Deleting a vertex of minimum degree $\delta(G)$ cannot reduce $d(G)$.

2. A ***triangle-free*** graph is one that contains no triangles C_3. Let G be a triangle-free graph.
(**a**) For each edge $xy \in E(G)$, show that $d(x) + d(y) \leq v(G)$.
(**b**) Deduce that $\sum_{v \in V(G)} \big(d(v)\big)^2 \leq e(G)v(G)$.
(**c**) Using the Cauchy-Schwarz inequality, establish that $e(G) \leq \frac{1}{4}\big(v(G)\big)^2$.

3. Let P and Q be paths of maximum length in a connected graph G. Prove that P and Q have a common vertex.

4. Two Eulerian tours are ***equivalent*** if they have the same unordered pairs of consecutive edges, viewed cyclically (the starting point and direction are unimportant). A cycle, for example, has only one equivalence class of Eulerian tours. How many equivalence classes of Eulerian tours are there in the graph below?



5. The Petersen graph is the Kneser graph $KG_{5,2}$; this means that it has a vertex for each 2-element subset of $\{1,2,3,4,5\}$ and two vertices are adjacent if and only if the corresponding 2-element subsets are disjoint.
(**a**) If two vertices are nonadjacent in the Petersen graph, then prove they have exactly one common neighbour.
(**b**) Show that the minimal length of a cycle in the Petersen graph is 5.
(**c**) Prove that the Petersen graph has no cycle of length 7.