Problem Set #5

Due: Thursday, 11 October 2012

Students registered in MATH 401 should submit solutions to three of the following problems. Students in MATH 801 should submit solutions to all five.

1. Let C be a cycle in a connected weighted graph, and let e be an edge of maximum weight on C. Prove that there is an optimal spanning tree (a.k.a. minimal weight spanning tree) not containing e. Using this prove that iteratively deleting a heaviest non-cut-edge until the remaining graph is acyclic produces an optimal spanning tree.

2. Assign integer weights to the edges of K_n. Let the weight of a cycle be the sum of the weights of its edges. Prove that all cycles have even weight if and only if the subgraph formed by edges with odd weight is a spanning complete bipartite subgraph.

Hint. Show that every component of the subgraph consisting of the edges with even weight is a complete graph.

3. (**a**) Let B be a block of a graph G and let P be a path in G connecting two vertices of B. Show that P is contained in B.
(**b**) Deduce that a spanning subgraph T of a connected graph G is a spanning tree if and only if $T \cap B$ is a spanning tree of B for every block B of G.

4. (**a**) Prove that two distinct edges lie in the same block of a graph if and only if they belong to a common cycle.
(**b**) Let e, f, and g be distinct edges in a graph G. Suppose that the graph G has a cycle through e and f and a cycle through f and g. Prove that G also has a cycle through e and g.

5. If the connected graph G is not a block, then prove that G has at least two blocks each of which contains exactly one cut vertex of G.