Problems 3

Due: Friday, 17 February 2023 before 17:00 EST

Students registered in MATH 413 should submit solutions to any three problems, whereas students in MATH 813 should submit solutions to all five.

P3.1. Let $I := \langle xz - y^2, wz - xy, wy - x^2 \rangle$ be an ideal in the polynomial ring $\mathbb{Q}[w, x, y, z]$.

- **(i)** Find (without using computer software) the reduced Gröbner basis of I with respect to the graded reverse lexicographic order and $w > x > y > z$.
- **(ii)** Find (without using computer software) the reduced Gröbner basis of I with respect to the lexicographic order and $w > x > y > z$.
- **(iii)** (*Bonus*) The ideal I has 8 distinct leading term ideals. Can you exhibit these eight monomial ideals?

P3.2. Fix the lexicographic order on the ring $S := \mathbb{K}[x_1, x_2, \dots, x_n]$ where $x_1 > x_2 > \cdots > x_n$. Let $\mathbf{A} := [a_{j,k}]$ be an $(m \times n)$-matrix with entries in the field $\mathbb{K}$. For all $1 \leqslant j \leqslant m$, let $f_j := a_{j,1}\, x_1 + a_{j,1}\, x_2 + \cdots + a_{j,n}\, x_n$ be the linear polynomial determined by the j-th row of the matrix $\mathbf{A}$. Suppose that $\mathbf{B}$ is the row-reduced echelon matrix associated to $\mathbf{A}$ and let $g_1, g_2, \dots, g_r$ be the linear polynomials determined by the nonzero rows in $\mathbf{B}$.

- **(i)** Prove that $\langle f_1, f_2, \dots, f_m \rangle = \langle g_1, g_2, \dots, g_r \rangle$.
- **(ii)** Show that $g_1, g_2, \dots, g_r$ form a Gröbner basis of the ideal $\langle f_1, f_2, \dots, f_m \rangle$.
- **(iii)** Explain why $g_1, g_2, \dots, g_r$ is the reduced Gröbner basis.

P3.3. Suppose we have numbers a, b, c which satisfy the equations

$$a + b + c = 2\,, \qquad a^2 + b^2 + c^2 = 18\,, \qquad \text{and} \qquad a^3 + b^3 + c^3 = 5\,.$$

- **(i)** Prove that $a^4 + b^4 + c^4 = 106$.
- **(ii)** Show that $a^5 + b^5 + c^5 \neq 17$.
- **(iii)** What are $a^5 + b^5 + c^5$ and $a^6 + b^6 + c^6$?

P3.4. The ***Whitney umbrella surface*** is the image of the polynomial map $\rho \colon \mathbb{A}^2 \to \mathbb{A}^3$ defined by

$$(u, v) \mapsto (uv, v, u^2)\,.$$

- **(i)** Find the equation(s) for the smallest algebraic subvariety in $\mathbb{A}^3$ containing the Whitney umbrella.
- **(ii)** Show that the parametrization fills up this algebraic subvariety over $\mathbb{C}$ but not over $\mathbb{R}$. Over $\mathbb{R}$, exactly what points are omitted?
- **(iii)** Show that the parameters u and v are not always uniquely determined by a point in $\mathbb{A}^3$. Find the points where uniqueness fails.

P3.5. Consider the ideal $I := \langle x^2 + 2y^2 - 12, x^2 + xy + y^2 - 12 \rangle$ in $\mathbb{Q}[x, y]$.

- **(i)** Find Gröbner basis for $I \cap \mathbb{Q}[x]$ and $I \cap \mathbb{Q}[y]$.
- **(ii)** Find all solutions to the equations $x^2 + 2y^2 = 12$ and $x^2 + xy + y^2 = 12$ in $\mathbb{C}$.
- **(iii)** Which of the solutions are rational?
- **(iv)** What is the smallest field $\mathbb{K}$ containing $\mathbb{Q}$ such that all solutions lie in $\mathbb{K}$?