Problems 3

Due: Monday, 24 October 2022 before 17:00 EDT

P3.1 For any two morphism $\varphi\colon B \to C$ and $\varphi'\colon B \to C$ of R-complexes, consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longleftarrow & B[1] & \xleftarrow{\;\delta\;} & \mathrm{Cone}(\varphi) & \xleftarrow{\;\rho\;} & C & \longleftarrow & 0 \\ & & {\scriptstyle \mathrm{id}^{B[1]}}\big\downarrow & & \big\downarrow{\scriptstyle \psi} & & \big\downarrow{\scriptstyle \mathrm{id}^{C}} & & \\ 0 & \longleftarrow & B[1] & \xleftarrow{\;\delta'\;} & \mathrm{Cone}(\varphi') & \xleftarrow{\;\rho'\;} & C & \longleftarrow & 0\,. \end{array}$$

whose rows are the canonical short exact sequences. Prove that the morphisms φ and φ' are homotopic if and only if there exists an isomorphism $\psi\colon \mathrm{Cone}(\varphi) \to \mathrm{Cone}(\varphi')$ that makes the diagram commute.

P3.2 Let $\beta\colon B' \to B$, $\beta'\colon B' \to B$, $\gamma\colon C \to C'$, and $\gamma'\colon C \to C'$ be commutative homomorphisms of R-complexes.

(i) Show that the homomorphism $\mathrm{Hom}(\beta,\gamma)\colon \mathrm{Hom}(B,C) \to \mathrm{Hom}(B',C')$ of $\Bbbk$-complexes is commutative.

(ii) When β or γ is null-homotopic, show that $\mathrm{Hom}(\beta,\gamma)$ is also null-homotopic.

(iii) When $\beta \sim \beta'$ and $\gamma \sim \gamma'$, show that $\mathrm{Hom}(\beta,\gamma) \sim \mathrm{Hom}(\beta',\gamma')$.

P3.3 Let B and C be R-complexes. For any integer k, prove that the composite homomorphism

$$\mathrm{Hom}(\mathrm{id}^{B[k],B},C)\,\mathrm{id}^{\mathrm{Hom}(B,C)[-k],\mathrm{Hom}(B,C)}\colon\ \mathrm{Hom}(B,C)[-k] \to \mathrm{Hom}(B[k],C)$$

is an isomorphism of $\Bbbk$-complexes and it is natural in B and C.

P3.4 Let B be an R^{o}-complex and let C be an R-complex. Set B^{o} to be the corresponding R-complex and C^{o} to be corresponding R^{o}-complex and consider the homomorphism

$$\varsigma^{B,C}\colon B \otimes C \to C^{\mathrm{o}} \otimes B^{\mathrm{o}}$$

of $\Bbbk$-complexes having degree 0 defined, for any integers i and j, any $b \in B_i$, and any $c \in C_j$, by $\varsigma^{B,C}(b \otimes c) = (-1)^{ij} c \otimes b$. Prove that $\varsigma^{B,C}$ is an isomorphism.