Problems 4

Due: Monday, 7 November 2022 before 17:00 EST

P4.1 Let A be an R-complex. For any R-complex B, let

$$\varsigma^B := \operatorname{Hom}(A, \operatorname{id}^{B,B[1]})\, \operatorname{id}^{\operatorname{Hom}(A,B)[1],\operatorname{Hom}(A,B)} \colon \operatorname{Hom}(A,B)[1] \to \operatorname{Hom}(A,B[1])$$

denote the natural isomorphism; compare with Problem 3.3. For any morphism $\varphi\colon B \to C$ of R-complexes, consider the homomorphism $\kappa\colon \operatorname{Cone}\big(\operatorname{Hom}(A,\varphi)\big) \to \operatorname{Hom}\big(A, \operatorname{Cone}(\varphi)\big)$ defined, for all homogeneous $\beta \in \operatorname{Hom}(A,B)[1]$ and all homogeneous $\gamma \in \operatorname{Hom}(A,C)$, by

$$\kappa\left(\begin{bmatrix}\beta\\ \gamma\end{bmatrix}\right) = \begin{bmatrix}\varsigma^B(\beta)\\ \gamma\end{bmatrix}.$$

Demonstrate that the κ is an isomorphism.

P4.2 A *semi-free filtration* of an R-complex F is an increasing sequence

$$0 \subseteq F^0 \subseteq F^2 \subseteq \cdots \subseteq F^{m-1} \subseteq F^m \subseteq \cdots$$

of subcomplexes such that $F = \bigcup_{m\in\mathbb{Z}} F^m$, $F^{-1} = 0$, and the underlying $\mathbb{Z}$-graded R-module $\bigoplus_{i\in\mathbb{Z}}(F^m/F^{m-1})_i$ for each succesive quotient R-complex F^m/F^{m-1} has a homogeneous basis of cycles. Prove that an R-complex is semi-free if and only if it admits a semi-free filtration.

P4.3 Let P be a semi-projective R-complex and let $0 \longleftarrow P \longleftarrow C \longleftarrow B \longleftarrow 0$ be a short exact sequence of R-complexes. Prove that the R-complex C is semi-projective if and only if the R-complex B is semi-projective.

Hint. Use Theorem 5.1.6 (g).

P4.4 Let $\psi\colon A \to B$, $\varphi\colon B \to C$, and $\theta\colon C \to D$ be morphisms of R-complexes. When $\varphi\,\psi$ and $\theta\,\varphi$ are homotopy equivalences, prove that ψ, φ, θ, and $\theta\,\varphi\,\psi$ are also homotopy equivalences.