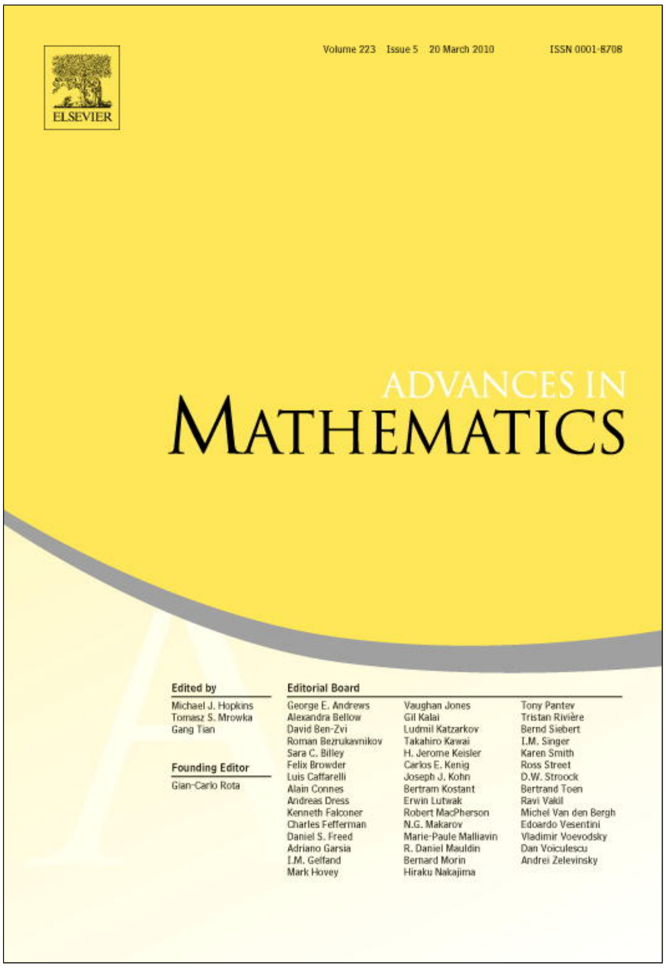
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Smooth and irreducible multigraded Hilbert schemes

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Abstract

The multigraded Hilbert scheme parametrizes all homogeneous ideals in a polynomial ring graded by an abelian group with a fixed Hilbert function. We prove that any multigraded Hilbert scheme is smooth and irreducible when the polynomial ring is $\mathbb{Z}[x, y]$, which establishes a conjecture of Haiman and Sturmfels.

Keywords: Hilbert schemes; Multigraded rings; Combinatorial commutative algebra

1. Introduction

Hilbert schemes are the fundamental parameter spaces in algebraic geometry. Multigraded Hilbert schemes, introduced in [12], consolidate many types of Hilbert schemes including Hilbert schemes of points in affine space, toric Hilbert schemes, G-Hilbert schemes for abelian G, and the original Grothendieck Hilbert scheme. The collection of all multigraded Hilbert schemes contains many well-documented pathologies. In contrast, this paper identifies a surprisingly large subcollection of multigraded Hilbert schemes that are both smooth and irreducible.

To be more explicit, let S be a polynomial ring over $\mathbb{Z}$ that is graded by an abelian group A. A homogeneous ideal $I \subseteq S$ is admissible if, for all $a \in A$, the $\mathbb{Z}$-module $(S/I)_a = S_a/I_a$ is a locally free with constant finite rank on $\operatorname{Spec}(\mathbb{Z})$. The Hilbert function $h_{S/I} : A \to \mathbb{N}$ is defined by $h_{S/I}(a) := \operatorname{rank}_{\mathbb{Z}}(S/I)_a$. Given $h : A \to \mathbb{N}$, Theorem 1.1 of [12] shows that there is a quasipro-

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jective scheme Hilb_S^h parametrizing all admissible S-ideals with Hilbert function h. Our main result is:

Theorem 1.1. *If $S = \mathbb{Z}[x, y]$ is graded by an abelian group A, then for any function $h : A \to \mathbb{N}$ the multigraded Hilbert scheme Hilb_S^h is smooth and irreducible.*

This theorem proves the conjecture in [12, Example 1.3] and [22, Conjecture 18.46]. Since $\mathrm{Spec}(\mathbb{Z})$ is the terminal object in the category of schemes, the theorem also extends, via base change, to the category of B-schemes where B is any irreducible scheme.

The hypothesis that S has two variables is essential in Theorem 1.1. Example 1.4 of [12] demonstrates that multigraded Hilbert schemes can be reducible when S has three variables. Even if one restricts to the standard $\mathbb{Z}$-grading, Theorem 1.2 of [3] shows that irreducibility fails; this also shows that the corank two result for toric Hilbert schemes [20, Theorem 1.1] does not extend to all multigraded Hilbert schemes. Remarkably, especially when compared with the connectedness of the Grothendieck Hilbert scheme [13, Corollary 5.9], Theorem 1 of [27] shows that multigraded Hilbert schemes can be disconnected. This evidence indicates that irreducibility of Hilb_S^h is rather exceptional.

Similarly, one does not expect a general multigraded Hilbert scheme Hilb_S^h to be smooth. Indeed, the philosophy in [29, §1.2] suggests that most multigraded Hilbert schemes contain complicated singularities. For example, Theorem 1.1 of [29] establishes that every singularity type of finite type over $\mathrm{Spec}(\mathbb{Z})$ appears on some Hilb_S^h when S has at least five variables. With this in mind, Theorem 1.1 provides a surprisingly comprehensive, but certainly not exhaustive, class of smooth and irreducible multigraded Hilbert schemes.

We were particularly inspired by [6], although each basic step in the proof of Theorem 1.1 has a counterpart in at least one of the following papers [13,7,16,25,23,11,21,15,20,24,8]. The basic steps in the proof are:

(i) We prove that either $\mathrm{Hilb}_S^h \cong \mathbb{P}^m \times \mathrm{Hilb}_S^{h'}$ or $\mathrm{Hilb}_S^h \cong \mathbb{A}^m \times \mathrm{Hilb}_S^{h'}$ where $\mathrm{Hilb}_S^{h'}$ parametrizes ideals with codimension greater than one.
(ii) We identify a distinguished point on Hilb_S^h and connect each point to this distinguished point by a rational curve.
(iii) We establish that the dimension of the tangent space is constant along these rational curves.
(iv) We show that the distinguished point on Hilb_S^h is nonsingular.

In all four steps, the combinatorial structure of the arguments allows us to work over an arbitrary field $\mathbb{k}$, so we are able lift our results to multigraded Hilbert schemes over $\mathbb{Z}$.

The first step, which appears in Section 2, shows that the multigraded Hilbert scheme Hilb_S^h parametrizing codimension-one ideals naturally splits into a product of a multigraded Hilbert scheme parametrizing equidimensional ideals of codimension one and a multigraded Hilbert scheme parametrizing ideals of higher codimension. This is tantamount to proving that there exists a functorial homogeneous factorization of the ideals with Hilbert function $h : A \to \mathbb{N}$. Among the papers listed above, only [7, §1] solves an analogous problem. Nevertheless, our factorization is striking because the primary decomposition of an ideal needed not be homogeneous when the grading group A has torsion; see [22, Example 8.10]. We establish this decomposition when S is a polynomial ring over $\mathbb{k}$ with an arbitrary number of variables. In the two variable case it plays a crucial role by reducing the proof of Theorem 1.1 to the study of schemes Hilb_S^h parametrizing ideals with finite colength. As a result, the remaining three steps assume that $S = \Bbbk[x, y]$ and $h : A \to \mathbb{N}$ has finite support.

In Section 3, we distinguish a point on Hilb_S^h by imposing a partial order on the set of all monomial ideals with Hilbert function $h : A \to \mathbb{N}$. The distinguished point corresponds to the maximum element in this poset, which we call the lex-most ideal. In the standard $\mathbb{Z}$-grading, the lex-most ideal coincides with the lex-segment ideal and corresponds to the lexicographic point on the Hilbert scheme. The larger class of lex-most ideals is required because lex-segment ideals do not necessarily exist for a general A-grading; see Example 3.13. In contrast with the standard-graded case, a lex-most ideal may not have extremal Betti numbers among all ideals with a given Hilbert function; see Example 3.14. The uniqueness of the lex-most ideal is the most novel aspect of the second step.

To complete the second step, we exhibit a chain of irreducible rational curves connecting each point on Hilb_S^h to the distinguished point. Each curve comes from the Gröbner degenerations of a binomial ideal. The binomial ideals, which are edge ideals in the sense of [1], arise from certain tangent directions. To designate a tangent direction, we use a combinatorial model for the tangent space to Hilb_S^h at a point corresponding to a monomial ideal. Our model extends the "cleft-couples" in [6, §2] and generalizes the "arrows" in [11, §2]. Unlike [21,24], we cannot restrict to Borel-fixed ideals because such ideals do not exist for arbitrary gradings. This approach has the advantage of proving that Hilb_S^h is rationally chain connected.

The third step, found in Section 4, identifies the tangent space to Hilb_S^h at each point along these rational curves with a linear subvariety of affine space. Finding the dimension of the tangent space is thereby equivalent to computing the rank of an explicit system of linear equations. Despite the conceptual simplicity, the inevitable combinatorial analysis is rather intricate. If we were working over an algebraically closed field of characteristic zero, then we could bypass this step by combining [17] and [7, Theorem 2.4]. Dealing with an explicit system of equations remarkably yields a higher level of generality.

For the fourth and final step, we demonstrate that the point on Hilb_S^h corresponding the lex-most ideal is nonsingular. This superficially resembles the smoothness of the lexicographic point in the original Grothendieck Hilbert scheme; see [26, Theorem 1.4]. From the previous step we know the dimension of the tangent space to Hilb_S^h at the distinguished point. To show that Hilb_S^h has the correct dimension at this point, it suffices to embed an affine space of the correct dimension into a neighborhood of the distinguished point. Following [6, Proposition 10], we achieve this in Section 5 by building an appropriate ideal that has the lex-most ideal as an initial ideal. The last section of the paper also contains the proof of Theorem 1.1.

Earlier work on the geometry of multigraded Hilbert schemes Hilb_S^h restricted either the possible grading groups A or the possible Hilbert functions $h : \mathbb{N} \to A$. In contrast, Theorem 1.1 limits only the number of variables. Indeed, our set-up deliberately includes gradings, called nonpositive [22, Definition 8.7], of S for which the grading group A has torsion or $\mathrm{rank}_{\mathbb{Z}} S_a = \infty$ for some $a \in A$. Unsurprisingly, the nonpositive gradings are the primary source of technical challenges. In fact, all four steps would be substantially easier if one excluded these cases.

Our success within this general framework leads to new questions: Can one characterize a larger collection of connected multigraded Hilbert schemes? When the polynomial ring S has more than two variables, does there exist a unique lex-most ideal? Do the maximal elements in the poset of monomial ideals with a given Hilbert function correspond to a nonsingular points?

1.1. Conventions

Throughout the paper, $\Bbbk$ is a field and $\mathbb{N}$ is the set of nonnegative integers. We write $\delta_{i,j}$ for the Kronecker delta: $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise. The lexicographic order on $\Bbbk[x, y]$ with $x > y$ is denoted by $>_+$ and the lexicographic order on $\Bbbk[x, y]$ with $x < y$ is denoted by $>_-$. For an ideal $I \subseteq \Bbbk[x, y]$, $\mathrm{in}_{>_+}(I)$ and $\mathrm{in}_{>_-}(I)$ are the initial ideals of I with respect to $>_+$ and $>_-$.

2. Factoring multigraded Hilbert schemes

We show in this section that the scheme Hilb^h_S naturally splits into a product of a multigraded Hilbert scheme parametrizing equidimensional codimension-one ideals and a multigraded Hilbert scheme parametrizing ideals with codimension greater than one. Let $\Bbbk$ be a field, let A be an abelian group, and let $S := \Bbbk[\boldsymbol{x}] = \Bbbk[x_1, \dots, x_N]$ be an A-graded polynomial ring with $N \geqslant 2$. Unlike the other sections, we do not assume that $N = 2$ in this section of the paper. We begin with a description of the multigraded Hilbert schemes parametrizing principal ideals.

Lemma 2.1. *Let $f \in S$ be a homogeneous polynomial of degree $d \in A$ such that the ideal $I := \langle f \rangle$ is admissible. If $h : A \to \mathbb{N}$ is the Hilbert function of S/I and $m := h(d)$, then*

$$\mathrm{Hilb}^h_S \cong \begin{cases} \mathbb{P}^m & \text{if } \dim_\Bbbk S_0 < \infty, \\ \mathbb{A}^m & \text{if } \dim_\Bbbk S_0 = \infty. \end{cases}$$

Proof. To begin, assume that $\dim_\Bbbk S_0 < \infty$. By [22, Theorem 8.6], we have that $\dim_\Bbbk S_a < \infty$ for all $a \in A$, and $S_0 = \Bbbk$. Thus the Hilbert function $h_S : A \to \mathbb{N}$ given by $h_S(a) = \dim_\Bbbk S_a$ is well-defined. Multiplication by f produces the short exact sequence

$$0 \to S(-d) \to S \to S/I \to 0,$$

which shows that $h(a) = h_S(a) - h_S(a - d)$. Since $S_0 = \Bbbk$, it follows that $h(d) = h_S(d) - 1$, so $\dim_\Bbbk(J_d) = 1$ for any ideal J with Hilbert function $h : A \to \mathbb{N}$. Applying this analysis to an element $g \in J_d$, we conclude that $J = \langle g \rangle$, so all ideals with Hilbert function $h : A \to \mathbb{N}$ are principal and generated in degree d. Hence, Hilb^h_S parametrizes the one-dimensional subspaces of S_d; in the language of [12, §3], the set $\{d\}$ is very supportive. Therefore, we have $\mathrm{Hilb}^h_S \cong \mathbb{P}^m$.

Secondly, assume that $\dim_\Bbbk S_0 = \infty$. The hypothesis that I is admissible places significant restrictions on S_0. Let $\boldsymbol{x}^{\boldsymbol{u}}$ be the initial term of f with respect to some monomial order on S. By [4, Theorem 15.3], the monomials not divisible by $\boldsymbol{x}^{\boldsymbol{u}}$ form a $\Bbbk$-basis for S/I. Since $I = \langle f \rangle$ is admissible, all but finitely many monomials in S_0 are divisible by $\boldsymbol{x}^{\boldsymbol{u}}$. It follows that S_0 has a homogeneous system of parameters consisting of a single element, so the Krull dimension of S_0 is 1. The ring S_0 is a normal semigroup ring by [22, p. 150], so we deduce that $S_0 = \Bbbk[\boldsymbol{x}^{\boldsymbol{v}}]$ for some monomial $\boldsymbol{x}^{\boldsymbol{v}} \in S$.

We next examine the S-module structure of the graded component S_d. Let $r \in \mathbb{N}$ be the largest nonnegative integer with $\boldsymbol{u} - r\boldsymbol{v} \in \mathbb{N}^N$ and set $\boldsymbol{w} := \boldsymbol{u} - r\boldsymbol{v}$. Let $\boldsymbol{x}^{\boldsymbol{w}'}$ be another monomial of degree d. Since $\dim_\Bbbk(S/I)_d < \infty$, all but finitely many monomials in S_d are divisible by $\boldsymbol{x}^{\boldsymbol{u}}$. Hence, the monomial $\boldsymbol{x}^{\boldsymbol{w}'+s\boldsymbol{v}}$ is divisible by $\boldsymbol{x}^{\boldsymbol{u}}$ for all $s \gg 0$, so for such s we have $\boldsymbol{w}'' := (\boldsymbol{w}' + s\boldsymbol{v}) - (\boldsymbol{w} + r\boldsymbol{v}) = (\boldsymbol{w}' - \boldsymbol{w}) + (s - r)\boldsymbol{v} \in \mathbb{N}^N$ with $\deg(\boldsymbol{x}^{\boldsymbol{w}''}) = 0$. Thus $\boldsymbol{w}'' = \ell\boldsymbol{v}$ for some $\ell \in \mathbb{N}$ and so $\boldsymbol{w}' - \boldsymbol{w}$ is a multiple of $\boldsymbol{v}$. By the construction of $\boldsymbol{w}$ this multiple must be nonnegative, so S_d has $\Bbbk$-basis $\{\boldsymbol{x}^{\boldsymbol{w}+s\boldsymbol{v}}\colon s \in \mathbb{N}\}$.

Finally, any monomial ideal with Hilbert function h must contain $\boldsymbol{x}^{\boldsymbol{w}+m\boldsymbol{v}}$. Since $\langle \boldsymbol{x}^{\boldsymbol{u}} \rangle$ has Hilbert function h, we have $r = m$ and there is only one monomial ideal with this Hilbert function; in the language of [12, §3] the set $\{d\}$ is very supportive. Therefore, Hilb_S^h parametrizes the ideals of the form $\langle \boldsymbol{x}^{\boldsymbol{u}} + c_1\boldsymbol{x}^{\boldsymbol{u}-\boldsymbol{v}} + \cdots + c_m\boldsymbol{x}^{\boldsymbol{u}-m\boldsymbol{v}} \rangle$ where $c_j \in \Bbbk$, so $\mathrm{Hilb}_S^h \cong \mathbb{A}^m$. $\square$

The next lemma contains the necessary algebraic preliminaries for factoring multigraded Hilbert schemes. The proof is complicated by our need to work over polynomial rings with coefficients in an arbitrary Noetherian $\Bbbk$-algebra.

Lemma 2.2. *Let K be a Noetherian $\Bbbk$-algebra, let $R := K \otimes_{\Bbbk} S$ be the A-graded polynomial ring with coefficients in K, and let I be a R-ideal. If J is the intersection of the codimension-one primary components of I and $Q := (I : J)$, then J and Q are homogeneous, J is a locally principal K-module, and $I = JQ$. Moreover, if* $\mathrm{Spec}(K)$ *is connected and I is admissible, then both J and Q are admissible ideals.*

Remark 2.3. The empty intersection of ideals equals R by convention, and $N \geqslant 2$, so $J \neq 0$. If K is a unique factorization domain, then J is simply generated by a greatest common divisor of any generating set for I. This follows from observation that in a unique factorization domain a primary ideal whose radical has codimension one is principal.

Proof of Lemma 2.2. We first show that J is homogeneous. Since we may assume that $\deg : \mathbb{N}^N \to A$ is surjective, the structure theorem for finitely generated abelian groups implies that $A \cong \mathbb{Z}^r \oplus \mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_s\mathbb{Z}$. It suffices to show that J is homogeneous with respect to each summand of A. The codimension-one primary components of I are homogeneous with respect a torsion-free grading by [2, IV, §3.3, Proposition 5], so J is also homogeneous with respect to a torsion-free grading. The case $A = \mathbb{Z}/m\mathbb{Z}$ remains. Consider an integral extension $\Bbbk'$ of the field $\Bbbk$ containing an mth root of unity ω, and set $R' := \Bbbk' \otimes_{\Bbbk} R$. Let $I' := \Bbbk' \otimes_{\Bbbk} I$ and let J' be the codimension-one equidimensional component of I'. From the intrinsic descriptions $J = \{f \in R\colon \operatorname{codim}(I : f) \geqslant 2\}$ and $J' = \{f \in R'\colon \operatorname{codim}(I' : f) \geqslant 2\}$, we see that $J = R \cap J'$. Thus, it is enough to show that J' is homogeneous with respect to a $(\mathbb{Z}/m\mathbb{Z})$-grading.

To accomplish this, fix generators for J'. For a generator $f \in R'$, set $f = \sum_{a\in A} f_a$ where each f_a is homogeneous of degree $a \in A$. We may assume that the generating set for J' has been chosen so that f_a does not lie in J' if $f \neq f_a$ and $f_a \neq 0$. Consider the automorphism $\phi : R' \to R'$ defined by $\phi(x_i) = \omega^{\deg(x_i)} x_i$ for $1 \leqslant i \leqslant N$. Since ϕ permutes the set of codimension-one primary components of I', we have $\phi(J') = J'$. If $f_a \neq 0$, then $\omega^a f - \phi(f) = \sum_{a'\in A}(\omega^a - \omega^{a'}) f_{a'} \in J'$ has fewer homogeneous parts. Iterating this procedure, it follows that one of the nonzero f_a lies in J' which means that f is itself homogeneous. Therefore, J' has a homogeneous set of generators and J is homogeneous.

Next, consider $\mathfrak{p} \in \mathrm{Spec}(K)$ and let $k(\mathfrak{p}) := K_{\mathfrak{p}}/\mathfrak{p}K_{\mathfrak{p}}$ be the residue field at $\mathfrak{p}$. It follows from Remark 2.3 that $J \otimes_K k(\mathfrak{p})$ is generated by the greatest common divisor of a generating set for $I \otimes_K k(\mathfrak{p})$. Since the ideal $\mathfrak{p}R_{\mathfrak{p}}$ lies in the Jacobson radical of $R_{\mathfrak{p}} := R \otimes_K K_{\mathfrak{p}}$, Nakayama's Lemma implies that $J_{\mathfrak{p}} := J \otimes_K K_{\mathfrak{p}}$ is generated by a single element f, so the ideal J is a locally principal K-module and $f \notin \mathfrak{p}R_{\mathfrak{p}}$.

To complete the first part, we examine $Q := (I : J)$. Since I and J are homogeneous, the ideal Q is as well. To see that $I = JQ$, it suffices to regard these ideals as K-modules and work locally. Suppose that $\mathfrak{p} \in \operatorname{Spec}(K)$ and $I_\mathfrak{p} := I \otimes_K K_\mathfrak{p} = \langle f_1, \ldots, f_\ell \rangle$. Since $I_\mathfrak{p} \subseteq J_\mathfrak{p} = \langle f \rangle$, we must have $f_i = f f_i'$ for some $f_i' \in R_\mathfrak{p}$. If $g \in Q_\mathfrak{p} = (I_\mathfrak{p} : J_\mathfrak{p})$ then $fg = \sum g_i f_i$ for some $g_i \in R_\mathfrak{p}$, so $f(g - \sum g_i f_i') = 0$. Because f either generates a codimension-one ideal or is a unit, it is not a zerodivisor, so $h \in \langle f_1', \ldots, f_\ell' \rangle$. We conclude that $Q_\mathfrak{p} = \langle f_1', \ldots, f_\ell' \rangle$ and $I_\mathfrak{p} = J_\mathfrak{p} Q_\mathfrak{p}$.

It remains to show that J and Q are admissible ideals. Let $d := \deg(f) \in A$. Since the homogeneous generator f of $J_\mathfrak{p}$ is not zerodivisor, there is a short exact sequence of $K_\mathfrak{p}$-modules $0 \to (R_\mathfrak{p})_{a-d} \to (R_\mathfrak{p})_a \to (R_\mathfrak{p}/J_\mathfrak{p})_a \to 0$ for each $a \in A$. Since $f \notin \mathfrak{p} R_\mathfrak{p}$, this sequence shows that $\operatorname{Tor}^1_{R_\mathfrak{p}}(k(\mathfrak{p}), (R_\mathfrak{p}/J_\mathfrak{p})_a) = 0$. The surjection $(R_\mathfrak{p}/I_\mathfrak{p})_a \to (R_\mathfrak{p}/J_\mathfrak{p})_a$ of $K_\mathfrak{p}$-modules establishes that $(R_\mathfrak{p}/J_\mathfrak{p})_a$ is finitely presented. Hence, Corollary 2 to [2, II, §3.2, Proposition 5] implies that $(R_\mathfrak{p}/J_\mathfrak{p})_a$ is free as a $K_\mathfrak{p}$-module for all $a \in A$. Multiplication by f also produces the short exact sequence

$$0 \to (R_\mathfrak{p}/Q_\mathfrak{p})_{a-d} \to (R_\mathfrak{p}/I_\mathfrak{p})_a \to (R_\mathfrak{p}/J_\mathfrak{p})_a \to 0. \tag{2.3.1}$$

The admissibility of I guarantees that $(R_\mathfrak{p}/I_\mathfrak{p})_a$ is a finite rank free $K_\mathfrak{p}$-module for all $a \in A$. The sequence (2.3.1) splits, so $(R_\mathfrak{p}/Q_\mathfrak{p})_a$ is free $K_\mathfrak{p}$-module of finite rank and $(R_\mathfrak{p}/J_\mathfrak{p})_a$ has finite rank. Since rank is upper semicontinuous, $(R/I)_a$ has constant rank on $\operatorname{Spec}(K)$, and $\operatorname{Spec}(K)$ is connected, we conclude that $(R/Q)_a$ and $(R/J)_a$ have constant rank on $\operatorname{Spec}(K)$ for all $a \in A$. $\square$

Before factoring multigraded Hilbert schemes, we record a geometric observation.

Lemma 2.4. *Given a function $h : A \to \mathbb{N}$, there is a constant $c = c(h)$ such that, for each Noetherian $\Bbbk$-algebra K, every admissible ideal $I \subseteq S \otimes_\Bbbk K$ with Hilbert function h has codimension c.*

Proof. Let K be a Noetherian $\Bbbk$-algebra and let $I \subseteq S \otimes_\Bbbk K$ be an admissible ideal with Hilbert function $h : A \to \mathbb{N}$. By restricting to the torsion-free component of A and the induced Hilbert function, it is enough to prove the result when $A = \mathbb{Z}^r$. Suppose that $P \in \operatorname{Ass}(I)$. We first claim that $\mathfrak{p} := P \cap K$ is a minimal prime ideal in K. Since P is an associated prime of I, there exists $f \in R := S \otimes_\Bbbk K$ such that $P = (I : f)$, so $lf \in I$ for all $l \in \mathfrak{p}$. Since $(R_\mathfrak{p}/I_\mathfrak{p})_a$ is a free $K_\mathfrak{p}$-module for all $a \in \mathbb{Z}^r$, we have either $f/1 = 0$ or $l/1 = 0$ in $R_\mathfrak{p}/I_\mathfrak{p}$. The first possibility would contradict $\mathfrak{p} = (I : f) \cap K$, so there is $l' \in K \setminus \mathfrak{p}$ with $l'l = 0 \in K$. Hence all primes in $\operatorname{Spec}(K)$ contained in $\mathfrak{p}$ must contain l. Because l was an arbitrary element of $\mathfrak{p}$, we deduce that $\mathfrak{p}$ is a minimal prime.

The codimension of I in R is the minimum of the codimensions of prime ideals in R containing I. If P is a minimal prime ideal containing I, then $P \in \operatorname{Ass}(I)$. Since $\mathfrak{p} = P \cap K$ is minimal in $\operatorname{Spec}(K)$, all prime ideals in R contained in P also intersect K in $\mathfrak{p}$, so $\operatorname{codim}(P, R) = \operatorname{codim}(P_\mathfrak{p}, R_\mathfrak{p}) = \operatorname{codim}(P_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, k(\mathfrak{p})[\boldsymbol{x}])$ where $k(\mathfrak{p}) := K_\mathfrak{p}/\mathfrak{p}K_\mathfrak{p}$ is the residue field at $\mathfrak{p}$. Applying this to a prime ideal P satisfying $\operatorname{codim}(I, R) = \operatorname{codim}(P, R)$, we see that $\operatorname{codim}(I, R) = \operatorname{codim}(I_\mathfrak{p}, R_\mathfrak{p}) = \operatorname{codim}(I_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, k(\mathfrak{p})[\boldsymbol{x}])$. Since $I_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is an admissible ideal in $k(\mathfrak{p})[\boldsymbol{x}]$ with Hilbert function $h : A \to \mathbb{N}$, the proof reduces to the case in which K is a field.

In this case, we have $\operatorname{codim}(I, R) = \dim R - \dim I = N - \dim \operatorname{in}(I)$ for any monomial initial ideal $\operatorname{in}(I)$ of I. Therefore, it suffices to observe that the dimension of a monomial ideal M is determined by its Hilbert function with respect to a $\mathbb{Z}^r$-grading. For any $a \in \mathbb{Z}^r$, consider the function $\bar{h}_a : \mathbb{N} \to \mathbb{N}$ defined by $\bar{h}_a(n) := h(na)$. By combining Theorem 1 in [28] with an appropriate Stanley decomposition of M (cf. [19, §3]), we see that the function $\bar{h}_a$ agrees with a quasipolynomial of degree d_a for $n \gg 0$ and the dimension of M is $r + \max\{d_a\colon a \in \mathbb{Z}^r\}$. □

The following theorem is the key result in this section.

Theorem 2.5. *Let H be a connected component of Hilb_S^h. There exists a Hilbert function $h' : A \to \mathbb{N}$ such that H is isomorphic to $X \times H'$, where X is either $\mathbb{P}^m$ or $\mathbb{A}^m$ for some $m \in \mathbb{N}$, H' is a connected component of $\mathrm{Hilb}_S^{h'}$, and H' parametrizes admissible ideals with codimension greater than one.*

To establish this decomposition, we use the associated functors of points; see [5, §VI]. Let $\mathbf{h}_Z$ be the functor of points determined by a scheme Z. For a $\Bbbk$-algebra K, we have $\mathbf{h}_Z(K) := \mathrm{Hom}(\mathrm{Spec}(K), Z)$. From this point of view, a morphism of schemes $Z \to Z'$ is equivalent to a natural transformation $\mathbf{h}_Z \to \mathbf{h}_{Z'}$ of functors. Since the schemes in Theorem 2.5 are all locally Noetherian over $\Bbbk$, we may assume that their associated functors of points map from the category of Noetherian $\Bbbk$-algebras to the category of sets.

By definition [12, §1], the scheme Hilb_S^h represents the Hilbert functor $\mathbf{Hilb}_S^h$. Recall that a homogeneous ideal I in $K \otimes_\Bbbk S$ is admissible if, for all $a \in A$, the K-module $(K \otimes_\Bbbk S_a)/I_a$ is a locally free of constant rank on $\mathrm{Spec}(K)$. For a $\Bbbk$-algebra K, $\mathbf{Hilb}_S^h(K)$ is the set of all admissible ideals I in $K \otimes_\Bbbk S$ with Hilbert function $h : A \to \mathbb{N}$.

Proof of Theorem 2.5. Consider the ideal sheaf $\mathscr{I}$ on $H \times \mathbb{A}^N$ which defines the universal admissible family over H with Hilbert function $h : A \to \mathbb{N}$. If $\mathscr{I}$ is zero, then the theorem is trivially true, so we may assume that $\mathscr{I} \neq 0$. Let $\mathscr{J}$ be the intersection of the codimension-one primary components of $\mathscr{I}$. Since H is connected, Lemma 2.2 shows that $\mathscr{J}$ and $\mathscr{Q} := (\mathscr{I} : \mathscr{J})$ are admissible. Let $h' : A \to \mathbb{N}$ and $h'' : A \to \mathbb{N}$ be the Hilbert functions associated to $\mathscr{Q}$ and $\mathscr{J}$ respectively. Lemma 2.2 also shows that $\mathscr{J}$ is locally principal over H, so h'' is the Hilbert function of some principal S-ideal. The degree of the local generator for $\mathscr{J}$ is constant, because H is connected. By combining these observations with Lemma 2.1, we see that $X := \mathrm{Hilb}_S^{h''}$ is isomorphic to either $\mathbb{P}^m$ or $\mathbb{A}^m$ for an appropriate $m \in \mathbb{N}$.

We next define a natural transformation $\Phi : \mathbf{h}_H \to \mathbf{h}_X \times \mathbf{Hilb}_S^{h'}$. Let K be a Noetherian $\Bbbk$-algebra and set $R := K \otimes_\Bbbk S$. Given an R-ideal I corresponding to a K-valued point of H, there is a map $\mathrm{Spec}(K) \to H$ such that I is the pull-back of $\mathscr{I}$. Using this map to pullback $\mathscr{J}$ and $\mathscr{Q}$, we obtain ideals $J \in \mathbf{h}_X(K)$ and $Q \in \mathbf{Hilb}_S^{h'}(K)$. Set $\Phi(I) := (J, Q)$. Let H' be the connected component of $\mathrm{Hilb}_S^{h'}$ containing the image of Φ, so $\Phi : \mathbf{h}_H \to \mathbf{h}_X \times \mathbf{h}_{H'}$.

To construct the inverse of Φ, consider K-valued points of X and H' corresponding to R-ideals J' and Q' respectively. Our choice of Hilbert functions $h', h'' : A \to \mathbb{N}$ together with Lemma 2.4 show that Q' has codimension greater than one and J' has codimension at most one. The proof of Lemma 2.1 establishes that J' is a locally principal K-module, so $J'_\mathfrak{p} = \langle f' \rangle$ where f' is a homogeneous nonzerodivisor of degree $d \in A$. Set $I' := J'Q'$. We claim that $(I' : J') = Q'$. It suffices to regard these ideals as K-modules and work locally. Suppose that $Q'_\mathfrak{p} = \langle f_1, \ldots, f_\ell \rangle$, so that $I'_\mathfrak{p} = \langle f' f_1, \ldots, f' f_\ell \rangle$. If $g \in (I'_\mathfrak{p} : f')$ then $gf' = \sum g_i f' f_i$ for some $g_i \in R_\mathfrak{p}$, so $f'(g - \sum g_i f_i) = 0$ and thus $g \in Q'_\mathfrak{p}$. The other inclusion is immediate, so we have $(I' : J') = Q'$. Thus multiplication by f' gives the short exact sequence $0 \to (R_\mathfrak{p}/Q'_\mathfrak{p})_{a-d} \to (R_\mathfrak{p}/I'_\mathfrak{p})_a \to (R_\mathfrak{p}/J'_\mathfrak{p})_a \to 0$. It follows that I' is admissible with Hilbert function $h : A \to \mathbb{N}$. The map $(J', Q') \mapsto J'Q'$ then defines a natural transformation $\Psi : \mathbf{h}_X \times \mathbf{h}_{H'} \to \mathbf{Hilb}_S^h$. If I is a K-valued point of H, then Lemma 2.2 implies that $I = JQ$, where $\Phi(I) = (J, Q)$, so I lies in the image of Ψ. Therefore, the unique connected component of Hilb_S^h containing the image of Ψ is H, and $\Psi : \mathbf{h}_X \times \mathbf{h}_{H'} \to \mathbf{h}_H$.

To finish the proof, we observe that Φ and Ψ are mutually inverse. In the last paragraph we showed that $\Psi \circ \Phi$ is the identity on H, so it suffices to check that if J' and Q' correspond to K-valued points of X and H', then J' is the codimension-one equidimensional part of $J'Q'$. The fact that $Q' = (J'Q' : J')$ then follows as above. Again it suffices to work locally on $\mathrm{Spec}(K)$. Since $(I'_{\mathfrak{p}} : f') = Q'_{\mathfrak{p}}$ has codimension greater than one by Lemma 2.4, f' lies in the codimension-one part of $I'_{\mathfrak{p}}$. If f' did not generate $I'_{\mathfrak{p}}$ there would be a nonunit common divisor of every generator of $Q'_{\mathfrak{p}}$, which would contradict $\mathrm{codim}(Q'_{\mathfrak{p}}, R_{\mathfrak{p}}) > 1$. Hence, $\Phi \circ \Psi$ is the identity on $\mathbf{h}_X \times \mathbf{h}_{H'}$. $\square$

Example 2.6. Suppose that $A = \mathbb{Z}$, $S = \Bbbk[x, y]$, $\deg(x) = 1$ and $\deg(y) = -1$. Let $h : A \to \mathbb{N}$ be the Hilbert function of the ideal $I = \langle x^4y^3, x^3y^4, x^2y^5\rangle = \langle x^2y^3\rangle \cdot \langle x^2, xy, y^2\rangle$. Since Theorem 1.1 establishes that Hilb_S^h is irreducible, it follows from Theorem 2.5 that $\mathrm{Hilb}_S^h \cong \mathrm{Hilb}_S^{h''} \times \mathrm{Hilb}_S^{h'} \cong \mathbb{A}^2 \times \mathrm{Hilb}_S^{h'}$ where $h'' : A \to \mathbb{N}$ is the Hilbert function of the ideal $J = \langle x^2y^3\rangle$ and $h' : A \to \mathbb{N}$ is the Hilbert function of the ideal $Q = \langle x^2, xy, y^2\rangle$. Since S_{-1} has $\Bbbk$-basis $\{y, xy^2, x^2y^3, \ldots\}$, $\mathrm{Hilb}_S^{h''}$ parametrizes all ideals $\langle x^2y^3 + c_1xy^2 + c_2y\rangle$ with $c_1, c_2 \in \Bbbk$.

3. Rationally chain connected

In this section, we prove that Hilb_S^h is rationally chain connected when $\Bbbk$ is a field, $S = \Bbbk[x, y]$, and $|h| := \sum_{a \in A} h(a) < \infty$. Indeed, we show that there exists a distinguished monomial ideal in S, called the lex-most ideal, and a finite chain of irreducible rational curves on Hilb_S^h connecting any point to the point corresponding to this lex-most ideal. The key to exhibiting these curves is a combinatorial model for the tangent space to Hilb_S^h at a point corresponding to a monomial ideal.

Consider a monomial ideal M in S with Hilbert function $h : A \to \mathbb{N}$ and let $x^{p_0}y^{q_0}$, $x^{p_1}y^{q_1}, \ldots, x^{p_n}y^{q_n}$ be the minimal generators of M where $p_0 > \cdots > p_n \geqslant 0$ and $0 \leqslant q_0 < \cdots < q_n$. The ideal M has finite colength if and only if $p_n = 0 = q_0$. An ***arrow*** associated to M is a triple $(i, u, v) \in \mathbb{N}^3$ where $0 \leqslant i \leqslant n$, the monomial $x^{p_i}y^{q_i}$ is a minimal generator of M, and x^uy^v is a standard monomial for M with the same degree as $x^{p_i}y^{q_i}$. Because $x^uy^v \notin M$, we must have either $u < p_i$ or $v < q_i$. We visualize an arrow (i, u, v) as the vector $\left[\begin{smallmatrix} u-p_i \\ v-q_i \end{smallmatrix}\right]$ originating at the (p_i, q_i)-cell and terminating at the (u, v)-cell; see Fig. 1.

Remark 3.1. Despite similar nomenclature, our definition of an arrow is different from [11, Proposition 2.4], [15, §2] and [22, §18.2]. In these sources, an 'arrow' refers to an equivalence class of vectors; the equivalence relation arises from certain horizontal and vertical translations. By fixing the tails of our arrows at minimal generators of M, we are choosing elements in each equivalence class. The 'significant arrows' defined below are in bijection with the nonzero equivalence classes. This strategy follows [6, §2].

Arrows are classified by their direction and position of their head relative to M. To indicate the direction, we say that an arrow (i, u, v) is ***positive*** if $u > p_i$, ***nonnegative*** if $u \geqslant p_i$, ***nonpositive*** if $v \geqslant q_i$, or ***utterly insignificant*** if both $u < p_i$ and $v < q_i$. The second aspect of our classification is determined by the monomial x^uy^v which we regard as the head of the arrow (i, u, v). A nonnegative arrow (i, u, v) is ***significant*** if $i > 0$ and $x^{u+p_{i-1}-p_i}y^v \in M$. We denote

Fig. 1. Three significant arrows for $\langle x^4, x^2y, y^2\rangle$.

by $T_{\geqslant 0}(M)$ the set of all nonnegative significant arrows of M. The subset of $T_{\geqslant 0}(M)$ consisting of all positive significant arrows plays a central role and is denoted by $T_+(M)$. Similarly, we call a nonpositive arrow ***significant*** if $i < n$ and $x^u y^{v-q_i+q_{i+1}} \in M$, and denote by $T_{\leqslant 0}(M)$ the set of all nonpositive significant arrows of M. An arrow is simply ***significant*** if it belongs to $T(M) := T_{\geqslant 0}(M) \cup T_{\leqslant 0}(M)$, and ***insignificant*** otherwise. As the notation suggests, the significant arrows index a basis for the tangent space to Hilb_S^h at the point corresponding to M; see Section 4.

Remark 3.2. By definition, every utterly insignificant arrow is insignificant. If (i, u, v) is an utterly insignificant arrow, then we have $\deg(x^{p_i-u} y^{q_i-v}) = 0 \in A$, so $\dim_{\Bbbk} S_0 = \infty$. If (i, u, v) is an arrow with either $u = p_i$ or $v = q_i$, then either $v < q_i$ and $\deg(y^{q_i-v}) = 0 \in A$ or $u < p_i$ and $\deg(x^{p_i-u}) = 0 \in A$. In either case one variable has torsion degree, which also implies that $\dim_{\Bbbk} S_0 = \infty$.

Next, we associate an irreducible rational curve on Hilb_S^h to each positive significant arrow $\alpha := (k, \ell + p_k, m + q_k) \in T_+(M)$. To describe this curve, we define the α***-edge ideal*** to be

$$I_\alpha(t) := \langle x^{p_i} y^{q_i} \colon 0 \leqslant i < k \rangle + \langle x^{p_i} y^{q_i} - t x^{\ell+p_i} y^{m+q_i} \colon k \leqslant i \leqslant n \rangle, \tag{3.2.2}$$

where $t \in \Bbbk$. By construction, the S-ideal $I_\alpha(t)$ is homogeneous with respect to the A-grading and $M = I_\alpha(0)$. We occasionally regard $I_\alpha(t)$ as a family of ideals over the base $\mathbb{A}^1 = \mathrm{Spec}(\Bbbk[t])$.

Example 3.3. If $A = 0$ and $M = \langle x^4, x^2y, y^2\rangle$, then

$$\begin{aligned} T_{\geqslant 0}(M) &= \{(1,3,0), (1,2,0), (2,3,0), (2,2,0), (2,1,1), (2,0,1)\}, \\ T_{\leqslant 0}(M) &= \{(0,3,0), (0,2,0), (0,1,1), (0,0,1), (1,1,1), (1,0,1)\}, \\ T_+(M) &= \{(1,3,0), (2,3,0), (2,2,0), (2,1,1)\}. \end{aligned}$$

The insignificant arrows are $(0,0,0)$, $(0,1,0)$, $(1,0,0)$, $(1,1,0)$, $(2,0,0)$ and $(2,1,0)$. If $\alpha = (1,3,0) \in T_+(M)$ then $k = 1$, $\ell = 1$, $m = -1$ and $I_\alpha(t) = \langle x^4, x^2y - tx^3, y^2 - txy\rangle$. The arrows $(1,3,0) \in T_+(M)$, $(0,0,1) \in T_{\leqslant 0}(M)$ and $(2,0,1) \in T_{\geqslant 0}(M)$ are pictured in Fig. 1.

The next result justifies our choice of generators for $I_\alpha(t)$. We write $>_-$ for the lexicographic monomial order on $S = \Bbbk[x, y]$ with $x < y$, and $\delta_{i,j}$ is the Kronecker delta.

Lemma 3.4. *If $\alpha = (k, \ell + p_k, m + q_k) \in T_+(M)$, then the defining generators of $I_\alpha(t)$ form a minimal Gröbner basis with respect $>_-$ and $M = \mathrm{in}_{>_-}(I_\alpha(t))$. Moreover, there is an index σ such that $0 \leqslant \sigma < k$, $\ell + p_{k-1} \geqslant p_\sigma$, $m + q_k \geqslant q_\sigma$, and the syzygies of $I_\alpha(t)$ are generated by $y^{-q_{i-1}+q_i}\mathbf{e}_{i-1} - x^{p_{i-1}-p_i}\mathbf{e}_i - \delta_{i,k} t x^{\ell+p_{k-1}-p_\sigma} y^{m+q_k-q_\sigma}\mathbf{e}_\sigma$ for $1 \leqslant i \leqslant n$ where $\mathbf{e}_0, \dots, \mathbf{e}_n$ is the standard basis for the A-graded free S-module $\bigoplus_{i=0}^n S(-\deg(x^{p_i} y^{q_i}))$.*

Proof. Since the minimal generators of M are the initial terms with respect to $>_{-}$ of the defining generators of $I_\alpha(t)$, it suffices to show that these generators form a Gröbner basis. By Buchberger's criterion [4, Exercise 15.19], we need only prove that certain S-polynomials reduce to zero modulo the generators of $I_\alpha(t)$, namely those for pairs of generators corresponding to the minimal syzygies of M. For any monomial ideal in the ring $S = \Bbbk[x, y]$, Proposition 3.1 of [22] shows that the minimal syzygies correspond to adjacent pairs of minimal generators. The S-polynomial between any pair of monomials is always zero. For any pair of adjacent binomial generators in $I_\alpha(t)$, the S-polynomial is

$$y^{-q_{i-1}+q_i}\left(x^{p_{i-1}}y^{q_{i-1}} - tx^{\ell+p_{i-1}}y^{m+q_{i-1}}\right) - x^{p_{i-1}-p_i}\left(x^{p_i}y^{q_i} - tx^{\ell+p_i}y^{m+q_i}\right) = 0,$$

where $k < i \leqslant n$. Hence, the final S-polynomial to examine is

$$y^{-q_{k-1}+q_k}\left(x^{p_{k-1}}y^{q_{k-1}}\right) - x^{p_{k-1}-p_k}\left(x^{p_k}y^{q_k} - tx^{\ell+p_k}y^{m+q_k}\right) = tx^{\ell+p_{k-1}}y^{m+q_k}.$$

Since $\alpha \in T_+(M)$, we have $\ell > 0$ and $m < 0$, so the monomials $x^{p_i}y^{q_i}$ for $i \geqslant k$ cannot divide $x^{\ell+p_{k-1}}y^{m+q_k}$. However, $\alpha \in T_+(M)$ implies that $x^{\ell+p_{k-1}}y^{m+q_k} \in M$, so $x^{\ell+p_{k-1}}y^{m+q_k}$ is divisible by at least one of the monomials $x^{p_\sigma}y^{q_\sigma}$ for $\sigma < k$. Therefore, the final S-polynomial reduces to zero modulo the generators of $I_\alpha(t)$. The assertion about the syzygies of $I_\alpha(t)$ then follows from [4, Theorem 15.10]. $\square$

Example 3.5. If $A = 0$, $M = \langle x^4, x^2y, y^2\rangle$ and $\alpha = (1, 3, 0) \in T_+(M)$ as in Example 3.3, then the syzygies of the α-edge ideal $I_\alpha(t)$ are generated by $y\mathbf{e}_0 - x^2\mathbf{e}_1 - xt\mathbf{e}_0$ and $y\mathbf{e}_1 - x^2\mathbf{e}_2$; here $\sigma = 0$.

Example 3.6. If $A = 0$, $M = \langle x^7, x^6y, x^5y^2, x^4y^3, x^2y^4, y^6\rangle$, and $\alpha = (4, 3, 2) \in T_+(M)$, then the syzygies of the α-edge ideal $I_\alpha(t) := \langle x^7, x^6y, x^5y^2, x^4y^3, x^2y^4 - tx^3y^2, y^6 - txy^4\rangle$ are generated by $y\mathbf{e}_0 - x\mathbf{e}_1$, $y\mathbf{e}_1 - x\mathbf{e}_2$, $y\mathbf{e}_2 - x\mathbf{e}_3$, $y\mathbf{e}_3 - x^2\mathbf{e}_4 - t\mathbf{e}_2$ and $y^2\mathbf{e}_4 - x^2\mathbf{e}_5$; the index σ is 2.

Following [30, Théorème 3.2] (also see [6, Definition 17]), we introduce a partial order on the set of all monomial ideals with a given Hilbert function. Given two monomial ideals M and M' with the same Hilbert function, we say $M' \succcurlyeq M$ if, for all monomials $x^ry^s \in S$, the number of standard monomials for M' with degree equal to $\deg(x^ry^s)$ lexicographically less than or equal to x^ry^s is at least the number of standard monomials for M with degree equal to $\deg(x^ry^s)$ lexicographically less than or equal to x^ry^s. The reflexivity, antisymmetry and transitivity of $\succcurlyeq$ follow from the properties of the canonical order on $\mathbb{N}$. Given a Hilbert function $h : A \to \mathbb{N}$, let $\mathcal{P}_h$ denote the poset of all monomial ideals with Hilbert function h. If $M' \neq M$ and $M' \succcurlyeq M$, then we write simply $M' \succ M$.

Remark 3.7. Following [22, §3.1], we identify a monomial ideal M in $S = \Bbbk[x, y]$ with its staircase diagram. When the Hilbert function $h : A \to \mathbb{N}$ of M satisfies $|h| < \infty$, the staircase diagram of M is a Young diagram (in the French tradition). Hence, the rows of the diagram correspond to the parts of a partition of $|h|$. When $A = 0$, and $|h| < \infty$, the partial order $\succ$ is the dominance order applied to the conjugate partitions.

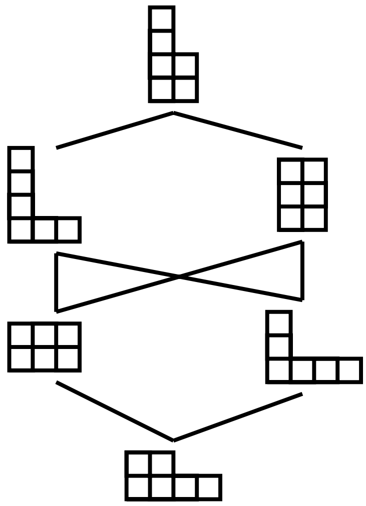

Fig. 2. Hasse diagram for the poset in Example 3.8.

Example 3.8. Suppose that $A = \mathbb{Z}$ and $\deg(x) = 1 = \deg(y)$. Among the eleven monomial ideals of colength six in S, there are exactly six monomial ideals with Hilbert function given by $h(0) = 1$, $h(1) = 2$, $h(2) = 2$, $h(3) = 1$ and $h(a) = 0$ for all $a \geqslant 3$. Fig. 2 illustrates the Hasse diagram for the poset $\mathcal{P}_h$.

The next lemma records a well-known geometric interpretation for Gröbner bases. We write $>_+$ for the lexicographic monomial order on $S = \Bbbk[x, y]$ with $x > y$.

Lemma 3.9. *Given an S-ideal I corresponding to a point on Hilb_S^h, the Gröbner degenerations of I with respect to $>_+$ and $>_-$ describe an irreducible rational curve on Hilb_S^h containing the points corresponding to I, $\mathrm{in}_{>_-}(I)$ and $\mathrm{in}_{>_+}(I)$.*

Proof. Proposition 15.16 in [4] gives a weight vector $w \in \mathbb{Z}^2$ such that $\mathrm{in}_w(I) = \mathrm{in}_{>_-}(I)$ and $\mathrm{in}_{-w}(I) = \mathrm{in}_{>_+}(I)$. Applying Theorem 15.17 in [4], we obtain a flat family of admissible ideals over $\mathbb{P}^1$ in which the fibers over 0, 1 and ∞ are $\mathrm{in}_{>_-}(I)$, I and $\mathrm{in}_{>_+}(I)$ respectively. Since Hilb_S^h is a fine moduli space, this family gives a map from $\mathbb{P}^1$ to Hilb_S^h whose image contains the points corresponding to I, $\mathrm{in}_{>_-}(I)$ and $\mathrm{in}_{>_+}(I)$. $\square$

We now apply Lemma 3.9 to describe the irreducible rational curve on Hilb_S^h associated to the positive significant arrow $\alpha \in T_+(M)$.

Proposition 3.10. *Let M be a monomial ideal in S. If $\alpha \in T_+(M)$ and $t \neq 0$, then $I_\alpha(t)$ has exactly two initial ideals, namely $M = \mathrm{in}_{>_-}(I_\alpha(t))$ and $M' := \mathrm{in}_{>_+}(I_\alpha(t))$. Moreover, we have $M' \succ M$ and, on Hilb_S^h, the points corresponding to M and M' lie on an irreducible rational curve.*

Proof. Let $\alpha = (k, \ell + p_k, m + q_k)$ and consider the vector $\left[\begin{smallmatrix} \ell \\ m \end{smallmatrix}\right] \in \mathbb{Z}^2$. By construction, the ideal $I_\alpha(t)$ is homogeneous with respect to the induced $(\mathbb{Z}^2/\mathbb{Z}\left[\begin{smallmatrix} \ell \\ m \end{smallmatrix}\right])$-grading of S. A polynomial in S that is homogeneous with respect to this grading has only two possible initial terms. Moreover, these two initial terms are given by $>_+$ and $>_-$. Hence, there are only two equivalence classes of monomial orders with respect to $I_\alpha(t)$. It follows that $I_\alpha(t)$ has at most two distinct initial ideals.

Lemma 3.4 shows that $M = \mathrm{in}_{>_-}(I_\alpha(t))$. Lemma 3.9 shows that the Gröbner degenerations of $I_\alpha(t)$ give with an irreducible rational curve on Hilb_S^h containing the points corresponding to M, $I_\alpha(t)$ and M', so it remains to show that $M' \succ M$.

Since $t \neq 0$, we know $I_\alpha(t) \neq M$, so $M = \mathrm{in}_{>_-}(I_\alpha(t))$ implies $M \neq M'$. Suppose that $M' \not\succ M$; this means there exists a monomial $x^r y^s \in S$ such that the number of standard monomials for M' with degree equal to $\deg(x^r y^s)$ lexicographically less than or equal to $x^r y^s$ is strictly less than the number of such standard monomials for M. Choosing $x^r y^s$ to be the lexicographically smallest monomial with this property guarantees that $x^r y^s \in M'$, $x^r y^s \notin M$ and each monomial lexicographically less than or equal to $x^r y^s$ with degree equal to $\deg(x^r y^s)$ is either in both of M' and M or in neither monomial ideal. Because $I_\alpha(t)$ is a binomial ideal, the remainder of $x^r y^s$ on division by the Gröbner basis for $I_\alpha(t)$ with respect to $>_+$ is a monomial, say $x^u y^v$. Since $M' = \mathrm{in}_{>_+}(I_\alpha(t))$, we have $x^u y^v \notin M'$, so $x^r y^s \neq x^u y^v$. Hence, $x^r y^s - t^\lambda x^u y^v \in I_\alpha(t)$ for some $\lambda > 0$ and $x^r y^s >_+ x^u y^v$ which implies that $x^u y^v \notin M$. But this means $\mathrm{in}_{>_-}(x^r y^s - t^\lambda x^u y^v) \notin M = \mathrm{in}_{>_-}(I_\alpha(t))$ which is a contraction. $\square$

Example 3.11. If $A = 0$, $M = \langle x^4, x^2y, y^2\rangle$ and $\alpha = (1,3,0) \in T_+(M)$ as in Example 3.3, then we have $I_\alpha(t) = \langle x^4, x^2y - tx^3, y^2 - txy\rangle$ and its initial ideals are $M = \mathrm{in}_{>_-}(I_\alpha(t))$ and $M' := \langle x^3, xy, y^4\rangle = \mathrm{in}_{>_+}(I_\alpha(t))$. The map $[z_0 : z_1] \mapsto \langle x^4, z_0x^2y - z_1x^3, z_0y^2 - z_1xy, y^4\rangle$ induces a morphism from $\mathbb{P}^1$ to the appropriate multigraded Hilbert scheme. In particular, we have $[1:0] \mapsto M$, $[0:1] \mapsto M'$, and $[1:t] \mapsto I_\alpha(t)$.

For a Hilbert function $h : A \to \mathbb{N}$ satisfying $|h| := \sum_{a\in A} h(a) < \infty$, $|h|$ equals the colength of the ideals parametrized by Hilb_S^h.

Proposition 3.12. *For a Hilbert function $h : A \to \mathbb{N}$ with $|h| < \infty$, there exists a unique monomial ideal $L_h \in \mathcal{P}_h$ such that $T_+(L_h) = \emptyset$. Thus, the poset $\mathcal{P}_h$ has a unique maximal element.*

We call the monomial ideal L_h of Proposition 3.12 the ***lex-most*** ideal with Hilbert function h.

Proof of Existence. Asserting $h : A \to \mathbb{N}$ is a Hilbert function means that there exists an ideal I with Hilbert function equal to h. Hence, $M = \mathrm{in}_{>_-}(I)$ is a monomial ideal with Hilbert function h. There are only finitely many monomial ideals with Hilbert function h, so the poset $\mathcal{P}_h$ has at least one maximal element. Proposition 3.10 shows that $M \in \mathcal{P}_h$ is not maximal when $T_+(M) \neq \emptyset$. Therefore, there is at least one monomial ideal $L_h \in \mathcal{P}_h$ with $T_+(L_h) = \emptyset$. $\square$

Proof of Uniqueness. We induct on $|h|$. Proposition 3.10 shows that only monomial ideals with no positive significant arrows can be maximal elements of $\mathcal{P}_h$. Suppose that the monomial ideal $M = \langle x^{p_0}y^{q_0}, \ldots, x^{p_n}y^{q_n}\rangle$ is a maximal element of $\mathcal{P}_h$, so $T_+(M) = \emptyset$. Since $|h| < \infty$, M has finite colength and $p_n = 0 = q_0$. If $|h| = 0$ or 1, then $\langle 1\rangle$ or $\langle x, y\rangle$ respectively is the unique monomial ideal in $\mathcal{P}_h$, so the base case of the induction holds.

For the induction step, we examine the ideal $(M : y)$. The minimal generators of $(M : y)$ are either $\langle x^{p_0}, x^{p_1}y^{q_1-1}, \ldots, y^{q_n-1}\rangle$ when $q_1 > 1$ or $\langle x^{p_1}, x^{p_2}y^{q_2-1}, \ldots, y^{q_n-1}\rangle$ when $q_1 = 1$. As a preamble, we prove that $T_+(M : y) = \emptyset$. If there exists a pair $(x^{p_i}y^{q_i-1}, x^u y^{v-1})$ corresponding to a positive significant arrow of $(M : y)$, then $i > 0$, $u - p_i > 0$, and $x^{u+p_{i-1}-p_i}y^{v-1} \in (M : y)$. The definition of the ideal quotient implies that $x^u y^v \notin M$, and $x^{u+p_{i-1}-p_i}y^v \in M$. Hence, $(i, u, v) \in T_+(M) = \emptyset$ which is a contradiction. Additionally, the short exact sequence

$$0 \to \frac{S}{(M:y)}\big(-\deg(y)\big) \xrightarrow{y} \frac{S}{M} \to \frac{S}{(x^{p_0}, y)} \to 0$$

implies that $|h'| = \sum_{a \in A} h'(a) = |h| - p_0 < |h|$ where $h' : A \to \mathbb{N}$ is the Hilbert function of $(M : y)$, so the induction hypothesis ensures that $(M : y)$ is unique. Therefore it suffices to show that all the maximal elements of $\mathcal{P}_h$ contain the same power of x as a minimal generator.

To complete the proof, we assume that M is chosen so that the power of the minimal generator x^{p_0} is maximal among all the maximal elements of $\mathcal{P}_h$. We break our analysis into two cases. First, suppose that there is no standard monomial $x^u y^v$ of M with degree equal to $\deg(x^{p_0-1}) - \deg(y)$ such that $x^u y^{v+1} \in M$. It follows that

$$\hbar := h\big(\deg\big(x^{p_0-1}\big)\big) - h\big(\deg\big(x^{p_0-1}\big) - \deg(y)\big)$$

is the number of standard monomials for M of degree $\deg(x^{p_0-1})$ that are pure powers of x. Moreover, x^{p_0-1} must be the $\hbar$th such monomial. For any $M' \in \mathcal{P}_h$, there must be at least $\hbar$ standard monomials of degree $\deg(x^{p_0-1})$ that are pure powers of x. As a result, x^{p_0-1} is standard for all M'. From our choice of M, we conclude that all the maximal elements of $\mathcal{P}_h$ contain x^{p_0} as a minimal generator in this case.

For the second case, suppose that there is a standard monomial $x^u y^v$ of M with degree equal to $\deg(x^{p_0-1}) - \deg(y)$ such that $x^u y^{v+1} \in M$. Since there exists a minimal generator of M dividing $x^u y^{v+1}$, there is an index $i > 0$ such that $p_i \leqslant u < p_{i-1}$ and $q_i = v + 1$. If $u < p_0 - 1$ then we have $(i, p_0 - 1 + p_i - u, 0) \in T_+(M) = \emptyset$ which is a contradiction. Hence, we may assume that $u = p_0 - 1$ which implies that $(v + 1)\deg(y) = \deg(y^{v+1}) = 0$. Now, consider a hypothetical monomial $x^r y^s \in S$ satisfying $\deg(x^r y^s) = \deg(x^{p_0-1})$ and $r < p_0 - 1$. Since the ideal M has finite colength, there is a $\zeta \geqslant v + 1$ such that $x^r y^\zeta \in M$ and $x^r y^{\zeta-1} \notin M$. Thus, there is $0 \leqslant \xi \leqslant v$ with $\deg(x^r y^s) = \deg(x^r y^{\zeta-\xi})$, because $\deg(y^{v+1}) = 0$. If $1 \leqslant j \leqslant n$ is the index such that $p_j \leqslant r < p_{j-1}$ and $\zeta = q_j$, then we have $\deg(x^{p_j} y^{q_j}) = \deg(x^{p_0-1-r+p_j} y^\xi)$, so $(j, p_0 - 1 - r + p_j, \xi) \in T_+(M) = \emptyset$ which is a contradiction. In other words, the hypothetical monomial $x^r y^s$ cannot exist. Since $h(\deg(x^{p_0-1})) > 0$, we deduce that x^{p_0-1} must be a standard monomial for all $M' \in \mathcal{P}_h$. From our choice of M, we again conclude that all the maximal elements of $\mathcal{P}_h$ contain x^{p_0} as a minimal generator in this case. □

Example 3.13. Suppose that $A = \mathbb{Z}/3\mathbb{Z}$, $\deg(x) = 1$ and $\deg(y) = 1$. The monomial ideals in S with Hilbert function $h(0) = 2$, $h(1) = 3$ and $h(2) = 1$ are $M := \langle x^5, xy, y^2\rangle$ and $M' := \langle x^2, xy, y^5\rangle$. The poset $\mathcal{P}_h$ is the chain $M' \succ M$. Since we have $x >_+ y^7$, $\deg(x) = \deg(y^7)$, $y^7 \in M'$ and $x \notin M'$, it follows that the lex-most ideal M' is not a lex-segment ideal. See [22, §2.4] for more information on lex-segment ideals.

Example 3.14. Suppose that $A = 0$. The monomial ideals in S with Hilbert function $h(0) = 3$ are $M := \langle x^3, y\rangle$, $M' := \langle x^2, xy, y^2\rangle$ and $M'' := \langle x, y^3\rangle$. The poset $\mathcal{P}_h$ is the chain $M'' \succ M' \succ M$. The monomial ideal M' has the largest Betti numbers rather than the maximal element M'' of $\mathcal{P}_h$. Thus the analogue of the Bigatti–Hulett Theorem [22, Theorem 2.24] is false.

We end this section with its central result. A scheme is rationally chain connected if two general points can be joined by a chain of irreducible rational curves; see [18, §IV.3].

Theorem 3.15. *If $S = \Bbbk[x, y]$ and the Hilbert function $h : A \to \mathbb{N}$ satisfies $|h| < \infty$, then the points on* Hilb^h_S *corresponding to monomial ideals are connected by irreducible rational curves associated to positive significant arrows. Consequently,* Hilb^h_S *is rationally chain connected.*

Proof. Consider a point on Hilb^h_S corresponding to a monomial ideal M. We first exhibit a finite chain of curves associated to positive significant arrows connecting the points on Hilb^h_S corresponding to M and L_h. If $M \neq L_h$, Proposition 3.12 implies that there exists $\alpha \in T_+(M)$. If $M' := \mathrm{in}_{>_-}(I_\alpha(t))$, then Proposition 3.10 produces an irreducible rational curve associated to α which contains the points corresponding M and M'. Proposition 3.10 also shows that $M' \succ M$ in $\mathcal{P}_h$. If $M' \neq L$, then we may repeat these steps. Since Proposition 3.12 shows that the lex-most ideal L_h is the unique maximal element in $\mathcal{P}_h$, this process terminates with a curve that contains the point corresponding to L_h. Thus, for every pair of points on Hilb^h_S corresponding to monomial ideals, there is a connected curve containing both points for which every irreducible component is a rational curve associated to a positive significant arrow.

For each closed point on Hilb^h_S, we produce an irreducible rational curve containing this point and a point corresponding to a monomial ideal. If the ideal I' corresponds to a point on Hilb^h_S, then Lemma 3.9 shows that the Gröbner degenerations of I' give an irreducible rational curve on Hilb^h_S which contains the points corresponding to I' and $\mathrm{in}_{>_-}(I')$. Therefore, for every pair of closed points on Hilb^h_S there is a connected curve, in which every irreducible component is rational, that contains both points. $\square$

4. Tangent spaces

This section relates the combinatorics of the significant arrows to the geometry of the multigraded Hilbert scheme. Given a monomial ideal M in $S = \Bbbk[x, y]$ with Hilbert function $h : A \to \mathbb{N}$ satisfying $|h| < \infty$, fix $\alpha \in T_+(M)$ and let $I_\alpha(t)$ be the α-edge ideal defined in (3.2.2). We prove that for all $t \in \Bbbk$ the significant arrows of M index a basis for the tangent space to Hilb^h_S at the point corresponding $I_\alpha(t)$. To accomplish this, we first identify the tangent space to Hilb^h_S at the point corresponding to $I_\alpha(t)$ with an explicit linear subspace.

Proposition 4.1. *Let $M = \langle x^{p_0} y^{q_0}, \dots, x^{q_n} y^{q_n} \rangle$ be a monomial ideal in S with Hilbert function $h : A \to \mathbb{N}$ and let $\alpha = (k, \ell + p_k, m + q_k)$ be a positive significant arrow for M. The tangent space to* Hilb^h_S *at the point corresponding to $I_\alpha(t)$ is isomorphic to the linear subspace of $\mathbb{A}^r :=$* $\mathrm{Spec}(\Bbbk[c^i_{u,v}\colon$ *(i, u, v) is an arrow of M*$])$ *cut out by the homogeneous linear equations*

$$F(i, u, v) := \sum_{\mu=0}^{b_{u,v}} t^\mu \big(c^{i-1}_{u-\mu\ell, v+q_{i-1}-q_i-\mu m} - c^i_{u-p_{i-1}+p_i-\mu\ell, v-\mu m} - \delta_{i,k} t c^\sigma_{u-p_{k-1}+p_\sigma-(\mu+1)\ell, v-q_k+q_\sigma-(\mu+1)m} \big), \tag{4.1.3}$$

where $1 \leqslant i \leqslant n$, $x^u y^v \notin M$, σ is the largest index satisfying $0 \leqslant \sigma < k$, $\ell + p_{k-1} \geqslant p_\sigma$ and $m + q_k \geqslant q_\sigma$, and $b_{u,v}$ is the largest nonnegative integer satisfying $x^{u-\kappa\ell} y^{v-\kappa m} \in M$ for all $0 < \kappa < b_{u,v}$.

Remark 4.2. The tangent space to Hilb^h_S at the point corresponding to $M = I_\alpha(0)$ is cut out by $F(i,u,v) = c^{i-1}_{u,v+q_{i-1}-q_i} - c^i_{u-p_{i-1}+p_i,v}$ for $1 \leqslant i \leqslant n$ and $x^u y^v \notin M$. Lemma 3.4 explains the importance of the index σ.

Proof of Proposition 4.1. For simplicity, set $I := I_\alpha(t)$. Lemma 3.4 shows that the given generators of I form a minimal Gröbner basis with respect to $>_{-}$, and Proposition 3.10 together with [4, Theorem 15.3] shows that the standard monomials of M form a $\mathbb{k}$-basis for S/I. By [12, Proposition 1.6], the tangent space is isomorphic to $(\mathrm{Hom}_S(I, S/I))_0$ where $0 \in A$. Given $\psi \in (\mathrm{Hom}_S(I, S/I))_0$, the ith generator of I maps to $\sum_{u,v} c^i_{u,v} x^u y^v$ where $c^i_{u,v} \in \mathbb{k}$ and the sum runs over all arrows for M of the form (i,u,v). If r_i is the number of such arrows and $r := \sum_{i=0}^n r_i$, then each $\psi \in (\mathrm{Hom}_S(M, S/M))_0$ produces a point $(c^i_{u,v}) \in \mathbb{A}^r$. Conversely, a point $(c^i_{u,v}) \in \mathbb{A}^r$ defines $\varphi \in (\mathrm{Hom}_S(\bigoplus_{i=0}^n S(-\deg(x^{p_i} y^{q_i})), S/I))_0$ by sending the ith standard basis element $\mathbf{e}_i$ of $\bigoplus_{i=0}^n S(-\deg(x^{p_i} y^{q_i}))$ to $\sum_{u,v} c^i_{u,v} x^u y^v$; again the sum runs over all arrows (i,u,v) of M. The syzygies of I determine whether φ restricts to $(\mathrm{Hom}_S(I, S/I))_0$. More precisely, Lemma 3.4 provides a free presentation of I having the form $S^n \xrightarrow{\partial} S^{n+1} \to I \to 0$. From this, we obtain the exact sequence

$$0 \to \mathrm{Hom}_S(I, S/I) \to \mathrm{Hom}_S\big(S^{n+1}, S/I\big) \xrightarrow{\bar{\partial}} \mathrm{Hom}_S\big(S^n, S/I\big),$$

so a point in $\mathbb{A}^r$ defines an element $(\mathrm{Hom}_S(I, S/I))_0$ if and only if $\bar{\partial}(\varphi) = \varphi \circ \partial = 0$. This condition is equivalent to a system of homogeneous linear equations in the $c^i_{u,v}$.

We next describe this system of equations. Since Lemma 3.4 provides a generating set of syzygies for I, we see that $\varphi \in (\mathrm{Hom}_S(I, S/I))_0$ if and only if we have

$$0 = y^{-q_{i-1}+q_i}\left(\sum_{u,v} c^{i-1}_{u,v} x^u y^v\right) - x^{p_{i-1}-p_i}\left(\sum_{u,v} c^i_{u,v} x^u y^v\right)$$
$$- \delta_{i,k} t x^{\ell+p_{k-1}-p_\sigma} y^{m+q_k-q_\sigma}\left(\sum_{u,v} c^\sigma_{u,v} x^u y^v\right) \in S/I,$$

where the sums run over all $x^u y^v \notin M$ and $c^i_{u,v} = 0$ when the triple (i,u,v) fails be an arrow. Lemma 3.4 also shows that the defining generators for I form a Gröbner basis such that $\mathrm{in}_{>_{-}}(I) = M$. By taking the normal form with respect to the generators of I, these equations produce an equation for each triple (i,u,v) such that $1 \leqslant i \leqslant n$ and $x^u y^v \notin M$. To be more explicit, observe that the coset in S/I containing $x^u y^v \notin M$ can have more than one element only when $x^u y^v \in \langle x^{\ell+p_j} y^{m+q_j} \colon j \geqslant k\rangle$. Let $b_{u,v}$ be the largest nonnegative integer such that $x^{u-\kappa\ell} y^{v-\kappa m} \in M$ for $0 < \kappa \leqslant b_{u,v}$. Since $\alpha \in T_+(M)$, we have $\ell > 0$, so $u - j\ell < 0$ for $j \gg 0$, and thus $b_{u,v} < \infty$. With this notation, the set $\{x^{u-\mu\ell} y^{v-\mu m} \colon 0 \leqslant \mu \leqslant b_{u,v}\}$ consists of all the monomials in S which reduce to $x^u y^v$ modulo the generators of I. Hence, the equation labelled by (i,u,v) is

$$F(i,u,v) := \sum_{\mu=0}^{b_{u,v}} t^\mu \big(c^{i-1}_{u-\mu\ell, v+q_{i-1}-q_i-\mu m} - c^i_{u-p_{i-1}+p_i-\mu\ell, v-\mu m}$$
$$- \delta_{i,k} t c^\sigma_{u-p_{k-1}+p_\sigma-(\mu+1)\ell, v-q_k+q_\sigma-(\mu+1)m}\big).$$

It follows that the tangent space is isomorphic to the linear subvariety of $\mathbb{A}^r$ cut out by these homogeneous linear equations. □

Example 4.3. If $A = 0$, $M = \langle x^4, x^2y, y^2\rangle$, and $\alpha = (1,3,0) \in T_+(M)$ as in Example 3.5, then the subspace in Proposition 4.1 is cut out by:

$$\begin{aligned}
&F(1,0,0) = 0, && F(1,1,0) = -tc^0_{0,0}, && F(1,2,0) = -c^1_{0,0} - tc^0_{1,0},\\
&F(1,3,0) = -c^1_{1,0} - tc^1_{0,1}, && F(1,0,1) = c^0_{0,0}, && F(1,1,1) = c^0_{1,0},\\
&F(2,0,0) = 0, && F(2,1,0) = 0, && F(2,2,0) = -c^2_{0,0},\\
&F(2,3,0) = tc^1_{2,0} + t^2c^1_{1,1} - c^2_{1,0} - tc^2_{0,1}, && F(2,0,1) = c^1_{0,0}, && F(2,1,1) = c^1_{1,0} + tc^1_{0,1}.
\end{aligned}$$

Example 4.4. As in Example 3.6, let $A = 0$, $M = \langle x^7, x^6y, x^5y^2, x^4y^3, x^2y^4, y^6\rangle$, and $\alpha = (4,3,2) \in T_+(M)$. Since M has six generators and colength 26, the linear subvariety in Proposition 4.1 is defined by $(6-1)(26) = 130$ equations. The following six equations illustrate some of the possibilities

$$\begin{aligned}
&F(1,5,0) = -c^1_{4,0}, && F(3,3,2) = c^2_{3,1} + tc^2_{2,3} + t^2c^2_{1,5} - c^3_{2,2} - tc^3_{1,4},\\
&F(2,4,1) = c^1_{4,0} - c^2_{3,1}, && F(5,0,5) = c^4_{0,3},\\
&F(4,2,3) = c^3_{2,2} - c^4_{0,3} - tc^2_{2,3}, && F(4,1,5) = c^3_{1,4} - tc^2_{1,5}.
\end{aligned}$$

We next describe the linear relations among the equations $F(i,u,v)$. By convention, we set $F(j,r,s) = 0$ if $r < 0$, $s < 0$ or $x^ry^s \in M$.

Lemma 4.5. *If $x^uy^v \notin M$ with $u < p_{i-1}$ and $v < q_i - q_{i-1}$, then we have the relation*

$$\begin{aligned}
0 = {}&\sum_{j=i}^{\sigma} F(j, u - p_{i-1} + p_{j-1}, v - q_i + q_j)\\
&+ \sum_{\lambda \geqslant 0}\sum_{j=\sigma+1}^{n} t^\lambda F(j, u - p_{i-1} + p_{j-1} - \lambda\ell, v - q_i + q_j - \lambda m).
\end{aligned} \tag{4.5.4}$$

Proof. We first consider the summands with $j \leqslant \sigma$. Since we have the inequalities $v < q_i$ and $v - q_i + q_j < q_j \leqslant q_\sigma \leqslant m + q_k$, the monomials $x^{u-p_{i-1}+p_{j-1}}y^{v-q_i+q_{j-1}}$ and $x^{u-p_{i-1}+p_j}y^{v-q_i+q_j}$ do not belong to $\langle x^{\ell+p_j}y^{m+q_j}\colon\ j \geqslant k\rangle$. Hence, we have

$$F(j, u - p_{i-1} + p_{j-1}, v - q_i + q_j) = c^{j-1}_{u-p_{i-1}+p_{j-1}, v-q_i+q_{j-1}} - c^j_{u-p_{i-1}+p_j, v-q_i+q_j}$$

so the first part of the relation (4.5.4) telescopes to

$$\sum_{j=i}^{\sigma} F(j, u - p_{i-1} + p_{j-1}, v - q_i + q_j) = -c^\sigma_{u-p_{i-1}+p_\sigma, v-q_i+q_\sigma},$$

because $c^{i-1}_{u,v-q_i+q_{i-1}} = 0$. Thus, it remains to analyze the variables $c^j_{r,s}$ in the double sum

$$\sum_{\lambda \geqslant 0} \sum_{j=\sigma+1}^{n} t^{\lambda} F(j, u - p_{i-1} + p_{j-1} - \lambda\ell, v - q_i + q_j - \lambda m). \tag{4.5.5}$$

To begin, we consider $j = n$. The only equations that might contain $c^n_{r,s}$ have the form $F(n, u - p_{i-1} + p_{n-1} - \lambda\ell, v - q_i + q_n - \lambda m)$; in this equation, such variables have the form $c^n_{u-p_{i-1}-(\mu+\lambda)\ell, v-q_i+q_n-(\mu+\lambda)m}$ for some $\mu \geqslant 0$ since $p_n = 0$. Since $u < p_{i-1}$ and $\ell > 0$, it follows that $u - p_{i-1} - (\mu + \lambda)\ell < 0$. Hence, no variable of the form $c^n_{r,s}$ appears in the double sum.

Next, suppose that $\sigma < j < n$. We first show that $c^j_{r,s}$ appears in at most two equations of the form (4.1.3). Specifically, if $x^r y^{s-q_j+q_{j+1}}$ reduces modulo $I_\alpha(t)$ to the standard monomial $x^{r+\mu\ell} y^{s-q_j+q_{j+1}+\mu m}$ for some $\mu \in \mathbb{N}$, then the variable $c^j_{r,s}$ appears in the equation $F(j + 1, r + \mu\ell, s - q_j + q_{j+1} + \mu m)$ with coefficient t^{μ}. Otherwise $x^r y^{s-q_j+q_{j+1}}$ reduces to zero modulo $I_\alpha(t)$ and the variable $c^j_{r,s}$ does not appear in an equation of the form $F(j + 1, r', s')$ for any $r', s' \in \mathbb{N}$. Similarly, if the monomial $x^{r+p_{j-1}-p_j} y^s$ reduces modulo $I_\alpha(t)$ to the standard monomial $x^{r+p_{j-1}-p_j+\mu'\ell} y^{s+\mu' m}$ for some $\mu' \in \mathbb{N}$, then $c^j_{r,s}$ appears in $F(j, r + p_{j-1} - p_j + \mu'\ell, s + \mu' m)$ with coefficient $-t^{\mu'}$. Otherwise $x^{r+p_{j-1}-p_j} y^s$ reduces to zero modulo $I_\alpha(t)$ and the variable $c^j_{r,s}$ does not appear in an equation of the form $F(j, r', s')$ for any $r', s' \in \mathbb{N}$. In summary, the variable $c^j_{r,s}$ appears in at most two equations of the form (4.1.3) and when it appears the coefficient is uniquely determined.

To complete this case, we show that if the variable $c^j_{r,s}$ appears in the double sum (4.5.5) then it appears twice: once with coefficient t^{ν} and once with coefficient $-t^{\nu}$. The equation $F(j+1, r+\mu\ell, s-q_j+q_{j+1}+\mu m)$ occurs in the double sum if and only if

$$\begin{bmatrix} r + \mu\ell \\ s - q_j + q_{j+1} + \mu m \end{bmatrix} = \begin{bmatrix} u - p_{i-1} + p_j - \lambda\ell \\ v - q_i + q_{j+1} - \lambda m \end{bmatrix} \quad \text{for some } \lambda \in \mathbb{N}.$$

Similarly, $F(j, r + p_{j-1} - p_j + \mu'\ell, s + \mu' m)$ occurs if and only if

$$\begin{bmatrix} r + p_{j-1} - p_j + \mu'\ell \\ s + \mu' m \end{bmatrix} = \begin{bmatrix} u - p_{i-1} + p_{j-1} - \lambda'\ell \\ v - q_i + q_j - \lambda' m \end{bmatrix} \quad \text{for some } \lambda' \in \mathbb{N}.$$

Rearranging these equations, it follows that $c^j_{r,s}$ appears in double sum only if

$$\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} u - p_{i-1} + p_j \\ v - q_i + q_j \end{bmatrix} - \nu \begin{bmatrix} \ell \\ m \end{bmatrix} \quad \text{for some } \nu \in \mathbb{N}; \tag{4.5.6}$$

either $\nu := \mu + \lambda$ and the coefficient of $c^j_{r,s}$ is t^{ν} or $\nu := \mu' + \lambda'$ and the coefficient of $c^j_{r,s}$ is $-t^{\nu}$. On the other hand, if (4.5.6) holds for some $\nu \in \mathbb{N}$, then we have

$$\begin{bmatrix} r \\ s - q_j + q_{j+1} \end{bmatrix} + \nu \begin{bmatrix} \ell \\ m \end{bmatrix} = \begin{bmatrix} u - p_{i-1} + p_j \\ v - q_i + q_{j+1} \end{bmatrix}$$

and

$$\begin{bmatrix} r + p_{j-1} - p_j \\ s \end{bmatrix} + \nu \begin{bmatrix} \ell \\ m \end{bmatrix} = \begin{bmatrix} u - p_{i-1} + p_{j-1} \\ v - q_i + q_j \end{bmatrix}.$$

Since $u < p_{i-1}$ and $v < q_i$, the monomials $x^{u-p_{i-1}+p_j} y^{v-q_i+q_{j+1}}$ and $x^{u-p_{i-1}+p_{j-1}} y^{v-q_i+q_j}$ do not belong to M, so both of the monomials $x^r y^{s-q_j+q_{j+1}}$ and $x^{r+p_{j-1}-p_j} y^s$ reduce modulo $I_\alpha(t)$ to standard monomials of M. Hence, the variable $c_{r,s}^j$ appears in twice in (4.5.5) with $\lambda := \nu - \mu \geqslant 0$ and $\lambda' := \nu - \mu' \geqslant 0$. We conclude that, when $c_{r,s}^j$ appears in the double sum, it appears twice with the same coefficient in t but with opposite signs.

Lastly, assume that $j = \sigma$. In this case, the variable $c_{r,s}^\sigma$ appears in at most three of the equations of the form (4.1.3); it could appear in

$$F(\sigma+1, u - p_{i-1} + p_\sigma - \mu\ell, v - q_i + q_{\sigma+1} - \mu m)$$

with coefficient t^μ for some $\mu \geqslant 0$, in $F(\sigma, u - p_{i-1} + p_{\sigma-1} - \mu'\ell, v - q_i + q_\sigma - \mu' m)$ with coefficient $-t^{\mu'}$ for some $\mu' \geqslant 0$, and in

$$F\big(k, u - p_{i-1} + p_{k-1} - p_\sigma - \big(\mu''+1\big)\ell, v - q_i + q_k - \big(\mu''+1\big)m\big)$$

with coefficient $-t^{\mu''+1}$ for some $\mu'' \geqslant 0$. As in the previous case, $c_{r,s}^\sigma$ appears in (4.5.5) if and only if (4.5.6) holds for some $\nu \geqslant 0$. However, only the first and third equation appear in the double sum, because the inner sum of (4.5.5) starts at $j = \sigma + 1$. As a consequence, if (4.5.6) holds with $\nu > 0$, then $c_{r,s}^j$ appears in (4.5.5) precisely twice with the same exponent on t but with opposite signs. Moreover, if (4.5.6) holds with $\nu = 0$, then $c_{r,s}^j$ appears in (4.5.5) only in the equation $F(\sigma+1, u - p_{i-1} + p_\sigma, v - q_i + q_{\sigma+1})$ with coefficient one. In summary, we have established that

$$c^\sigma_{u-p_{i-1}+p_\sigma, v-q_i+q_\sigma} = \sum_{\lambda \geqslant 0} \sum_{j=\sigma+1}^{n} t^\lambda F(j, u - p_{i-1} + p_{j-1} - \lambda\ell, v - q_i + q_j - \lambda m)$$

as required. $\square$

Using Lemma 4.5, we can describe the tangent space to Hilb_S^h at the point corresponding to $I_\alpha(t)$ by a smaller system of linear equations.

Corollary 4.6. *If M is a monomial ideal in $S = \Bbbk[x, y]$ with Hilbert function $h : A \to \mathbb{N}$ and $\alpha \in T_+(M)$, then the tangent space to Hilb_S^h at the point corresponding to $I_\alpha(t)$ is isomorphic to the subspace of $\mathbb{A}^r$ cut out by*

$$\mathcal{G} := \left\{ F(i,u,v)\colon \begin{array}{l} (i,u,v) \text{ is an arrow for } M \text{ with } 1 \leqslant i \leqslant n \text{ and} \\ \text{either } u \geqslant p_{i-1} \text{ or } v \geqslant q_i - q_{i-1} \end{array} \right\}.$$

Proof. Since Proposition 4.1 establishes that the tangent space is cut by all of the equations $F(i, u, v)$, it suffices to show that the $F(i, u, v)$ with $u < p_{i-1}$ and $v < q_i - q_{i-1}$ can be written as a linear combination of equations $F(i', u', v')$ not of this form. We induct on $q_i - q_{i-1} - v$. If $0 \geqslant q_i - q_{i-1} - v$, then the claim is vacuously true. Otherwise, consider the expression for $F(i, u, v)$ given by Lemma 4.5. For $j > i$, we have $v - q_i + q_j - \lambda m \geqslant q_j - q_{j-1}$, because $m < 0$ implies that $q_i - q_{j-1} + \lambda m \leqslant 0$. Hence, the only terms in this expression that might not be in $\mathcal{G}$ have the form $F(i, u - \lambda\ell, v - \lambda m)$. But these terms can be written as a linear combination of the elements of $\mathcal{G}$ by the induction hypothesis. $\square$

Example 4.7. If $A = 0$, $M = \langle x^4, x^2y, y^2\rangle$, and $\alpha = (1,3,0) \in T_+(M)$ as in Example 4.3, then Lemma 4.5 applied to $i = 1$ and $x, x^2, x^3 \notin M$ shows $0 = F(1,1,0) + tF(1,0,1)$, $0 = F(1,2,0) + F(2,0,1) + tF(1,1,1)$, and $0 = F(1,3,0) + F(2,1,1)$.

Example 4.8. As in Example 4.4, let $A = 0$, $M = \langle x^7, x^6y, x^5y^2, x^4y^3, x^2y^4, y^6\rangle$, and $\alpha = (4,3,2) \in T_+(M)$. For $i = 1$ and $x^5 \notin M$, Lemma 4.5 provides the relation

$$0 = F(1,5,0) + F(2,4,1) + F(3,3,2) + F(4,2,3) + F(5,0,5) + tF(3,2,4) + tF(4,1,5).$$

Since $x^2y^4 \in M$, we have $F(3,2,4) = 0$ by convention.

The following theorem is the essential result in this section. The proof shows that the dimension of the appropriate linear subspace of $\mathbb{A}^r$ equals the number of significant arrows.

Theorem 4.9. *Let M be a monomial ideal in $S = \Bbbk[x,y]$ with Hilbert function $h : A \to \mathbb{N}$ and fix $\alpha \in T_+(M)$. The significant arrows $T(M)$ of M index a basis for the tangent space to* Hilb_S^h *at the point corresponding to the edge ideal $I_\alpha(t)$ for all $t \in \Bbbk$.*

Proof. Let $I := I_\alpha(t)$ and $\alpha = (k, \ell + p_k, m + q_k) \in T_+(M)$. By Corollary 4.6, we see that the tangent space to Hilb_S^h at the point corresponding to I is isomorphic to the subspace of $\mathbb{A}^r$ cut out by those $F(i,u,v)$ where $x^uy^v \notin M$, $1 \leqslant i \leqslant n$, and either $u \geqslant p_{i-1}$ or $v \geqslant q_i - q_{i-1}$. It suffices to show that the insignificant arrows are in bijection with the initial terms (or leading variables) in this system of equations.

By definition, each $c_{u,v}^i$ corresponds to an arrow (i,u,v) associated to M. For convenience, we say that the variable $c_{u,v}^i$ is significant, nonnegative, etc., whenever same adjective applies to the corresponding arrow. Let $>_I$ be a monomial order on the polynomial ring $\Bbbk[c_{u,v}^i\colon (i,u,v) \text{ is an arrow of } M]$ satisfying the following conditions:

(C0) if $r - p_j > u - p_i$ then $c_{r,s}^j >_I c_{u,v}^i$;

(C1) for two nonnegative variables $c_{u,v}^i$ and $c_{r,s}^j$ with $r - p_j = u - p_i$, the inequality $i > j$ implies that $c_{u,v}^i >_I c_{r,s}^j$;

(C2) for two nonpositive variables $c_{u,v}^i$ and $c_{r,s}^j$ with $r - p_j = u - p_i$, the inequality $i < j$ implies that $c_{u,v}^i >_I c_{r,s}^j$;

(C3) for two utterly insignificant variables $c_{u,v}^i$ and $c_{r,s}^j$ with $r - p_j = u - p_i$, the inequality $i < j$ implies that $c_{u,v}^i >_I c_{r,s}^j$.

Observe that each equation $F(i,u,v)$ is homogeneous with respect to the grading defined by setting $\deg(c_{r,s}^j)$ equal to be the image of $\left[\begin{smallmatrix} r-p_j \\ s-q_j \end{smallmatrix}\right]$ in $\mathbb{Z}^2/\mathbb{Z}\left[\begin{smallmatrix} \ell \\ m \end{smallmatrix}\right]$. Hence, (C0) can be viewed as giving t a negative weight. Since $\ell > 0$, for each nonzero sum of the form $\sum_{\mu \geqslant 0} t^\mu c_{u'-\mu\ell, v'-\mu m}^j$, (C0) implies that $\mathrm{in}_{>_I}(\sum_{\mu \geqslant 0} t^\mu c_{u'-\mu\ell, v'-\mu m}^j) = t^{\widetilde{\mu}} c_{u'-\widetilde{\mu}\ell, v'-\widetilde{\mu}m}^j$, where $\widetilde{\mu} := \min\{\mu\colon c_{u'-\mu\ell, v'-\mu m}^j \neq 0\}$. It follows that

$$\begin{aligned} \mathrm{in}_{>_I}\bigl(F(i,u,v)\bigr) = \mathrm{in}_{>_I}\bigl(& t^{\widetilde{\mu}'} c_{u-\widetilde{\mu}'\ell, v+q_{i-1}-q_i-\widetilde{\mu}'m}^{i-1} - t^{\widetilde{\mu}} c_{u-p_{i-1}+p_i-\widetilde{\mu}\ell, v-\widetilde{\mu}m}^i \\ & - \delta_{i,k} t^{\widetilde{\mu}''+1} c_{u-p_{k-1}+p_\sigma-(\widetilde{\mu}''+1)\ell, v-q_k+q_\sigma-(\widetilde{\mu}''+1)m}^\sigma\bigr) \end{aligned}$$

for appropriate $\widetilde{\mu}', \widetilde{\mu}, \widetilde{\mu}'' \in \mathbb{N}$; (C0) also guarantees that the initial term is the variable accompanying the smallest exponent of t.

As a first step in constructing the bijection, we show that every insignificant arrow corresponds to the initial term of an element in $\mathcal{G}$. We divide the analysis into three cases.

Nonnegative case. If (i, u, v) is a nonnegative insignificant arrow, then $x^{u+p_{i-1}-p_i} y^v$ does not belong to M. Since $u \geqslant p_i$, we have $u + p_{i-1} - p_i \geqslant p_{i-1}$, so the variable $c^i_{u,v}$ appears as a nonzero term in $F(i, u + p_{i-1} - p_i, v) \in \mathcal{G}$ (i.e. $\widetilde{\mu} = 0$). Hence, (C0) and (C1) establish that $\mathrm{in}_{>_I}(F(i, u + p_{i-1} - p_i, v)) = c^i_{u,v}$.

Nonpositive case. If (i, u, v) is a nonpositive insignificant arrow, then $x^u y^{v-q_i+q_{i+1}}$ does not belong to M. Since $v \geqslant 0$, $v - q_i + q_{i+1} \geqslant -q_i + q_{i+1}$, the variable $c^i_{u,v}$ appears as a nonzero term in $F(i + 1, u, v - q_i + q_{i+1}) \in \mathcal{G}$ (so $\widetilde{\mu}' = 0$). Together (C0) and (C2) establish that $\mathrm{in}_{>_I}(F(i + 1, u, v - q_i + q_{i+1})) = c^i_{u,v}$.

Utterly insignificant case. If (i, u, v) is an utterly insignificant arrow, then we have $u < p_i \leqslant p_j$ for $j \leqslant i$ and $v < v - q_i + q_{i+1} < q_{i+1} \leqslant q_j$ for $j > i$, so $x^u y^{v-q_i+q_{i+1}} \notin M$. Hence, the variable $c^i_{u,v}$ appears as a nonzero term in $F(i + 1, u, v - q_i + q_{i+1}) \in \mathcal{G}$. Hence, (C0) and (C3) establish that $\mathrm{in}_{>_I}(F(i + 1, u, v - q_i + q_{i+1})) = c^i_{u,v}$.

By combining these three cases, we get an injective map from the insignificant arrows of M to the elements of $\mathcal{G}$.

To establish that this map is a bijection, we show that the initial term of each element of $\mathcal{G}$ corresponds to an insignificant arrow. Again, there are three cases. Fix $x^u y^v \notin M$.

Nonnegative case. If $u \geqslant p_{i-1}$, then we have the inequalities $u > u - p_{i-1} + p_i \geqslant p_i$. Hence, $(i, u - p_{i-1} + p_i, v)$ is an insignificant nonnegative arrow for M. Moreover, (C0) and (C1) ensure that $\mathrm{in}_{>_I}(F(i, u, v)) = c^i_{u-p_{i-1}+p_i,v}$.

Nonpositive case. If $v \geqslant q_i$, then we have the inequalities $v > v + q_{i-1} - q_i \geqslant q_{i-1}$. Hence, $(i - 1, u, v + q_{i-1} - q_i)$ is an insignificant nonnegative arrow for M. The inequality $u > u - p_{i-1} + p_i$ together with (C0) and (C2) imply that $\mathrm{in}_{>_I}(F(i, u, v)) = c^{i-1}_{u,v+q_{i-1}-q_i}$.

Utterly insignificant case. If $u < p_{i-1}$ and $q_i - q_{i-1} \leqslant v < q_i$, then we have the inequalities $\min(v, q_{i-1}) > v + q_{i-1} - q_i \geqslant 0$. Hence, $(i - 1, u, v + q_{i-1} - q_i)$ is an utterly insignificant arrow for M. The inequality $u > u - p_{i-1} + p_i$ together with (C0) and (C3) imply that $\mathrm{in}_{>_I}(F(i, u, v)) = c^{i-1}_{u,v+q_{i-1}-q_i}$.

In each case, the initial term of $F(i, u, v)$ is an insignificant arrow. Moreover, if we have $(i, u, v) \neq (j, r, s)$ with $F(i, u, v), F(j, r, s) \in \mathcal{G}$, then $F(i, u, v)$ and $F(j, r, s)$ have different initial terms. Therefore, we have a bijection between the insignificant arrows of M and the initial terms of the elements of $\mathcal{G}$.

Since the initial terms for the elements of $\mathcal{G}$ are relatively prime, they form a Gröbner basis with respect to $>_I$. Therefore, $T(M)$ indexes a basis for the tangent space to Hilb^h_S at the point corresponding to $I_\alpha(t)$ for all $t \in \Bbbk$. $\square$

Example 4.10. If $A = 0$, $M = \langle x^4, x^2y, y^2\rangle$, and $\alpha = (1, 3, 0) \in T_+(M)$ as in Example 4.7, then Theorem 4.9 shows that the tangent space to the appropriate multigraded Hilbert scheme at the point corresponding to $I_\alpha(t)$ is isomorphic to subspace cut out by

$$\begin{aligned}\langle \mathcal{G}\rangle &:= \bigl\langle F(1,0,1), F(1,1,1), F(2,2,0), F(2,3,0), F(2,0,1), F(2,1,1)\bigr\rangle \\ &= \bigl\langle c^0_{0,0}, c^0_{1,0}, -c^2_{0,0}, -c^2_{1,0} - tc^2_{0,1} + tc^1_{2,0} + t^2c^1_{1,1}, c^1_{0,0}, c^1_{1,0} + tc^1_{0,1}\bigr\rangle.\end{aligned}$$

Hence, the tangent space has dimension $(3)(6) - 6 = 12$.

5. Smoothness

The goal of this final section is to prove Theorem 1.1. To begin, we show that Hilb^h_S has at least one nonsingular point. This result parallels [26, Theorem 1.4] and our proof extends the techniques in [6, Proposition 10].

Proposition 5.1. *Let $S = \Bbbk[x, y]$ and let L_h be the lex-most ideal for a Hilbert function $h : A \to \mathbb{N}$ satisfying $|h| < \infty$. The ideal L_h corresponds to a nonsingular point on Hilb^h_S.*

Proof. Let d be the number of significant arrows associated to L_h. By Theorem 4.9, d equals the dimension of the tangent space to Hilb^h_S at the point corresponding to L_h. Thus, it suffices to show that the dimension of Hilb^h_S at this point is at least d. We accomplish this by constructing a map $\tau : \mathbb{A}^d \to \mathrm{Hilb}^h_S$ in which $\tau(0)$ corresponds to L_h and the dimension of the image is d. Since Hilb^h_S is a fine moduli space, the map $\tau : \mathbb{A}^d \to \mathrm{Hilb}^h_S$ is determined by an admissible ideal I in $R := K[x, y]$ where the coefficient ring is $K := \Bbbk[c^i_{u,v}\colon (i, u, v) \in T(M)]$. We may regard I as a family of ideals over the base $\mathbb{A}^d = \mathrm{Spec}(K)$.

We define the generators of I recursively. Since Proposition 3.12 states $T_+(L_h) = \emptyset$, the significant arrows associated to L_h are either nonpositive or have the form (i, p_i, v). For $1 \leqslant i \leqslant n$, consider $g_i := y^{-q_{i-1}+q_i} + \sum_{(i,p_i,v)\in T(L_h)} c^i_{p_i,v} y^{v-q_{i-1}}$. Since we have $(i, p_i, v) \in T(L_h)$, we see that $x^{p_{i-1}}y^v \in M$, so $v \geqslant q_{i-1}$ and g_i is a polynomial in R. Setting $f_n := \prod_{i=1}^n g_i$ means $\mathrm{in}_{>_+}(f_n) = \prod_{i=1}^n y^{-q_{i-1}+q_i} = y^{q_n} = x^{p_n}y^{q_n}$ because $p_n = q_0 = 0$. Next, suppose that the polynomials $f_{i+1}, \ldots, f_n$ are defined, with $\prod_{k=1}^j g_k$ dividing f_j for $i + 1 \leqslant j \leqslant n$. Given $(i, u, v) \in T_{\leqslant 0}(L_h)$, we have $x^u y^{v-q_i+q_{i+1}} \in L_h$, so the minimal monomial generator $x^{p_j}y^{q_j}$ divides $x^u y^{v-q_i+q_{i+1}}$ for some index j such that $i < j \leqslant n$. Let $\varepsilon = \varepsilon(i, u, v) := \max\{j\colon x^{p_j}y^{q_j} \text{ divides } x^u y^{v-q_i+q_{i+1}}\}$ and, for $0 \leqslant i < n$, define

$$f_i := \frac{1}{g_{i+1}}\left(x^{p_i-p_{i+1}} f_{i+1} + \sum_{(i,u,v)\in T_{\leqslant 0}(L_h)} c^i_{u,v} x^{u-p_\varepsilon} y^{v-q_i+q_{i+1}-q_\varepsilon} f_\varepsilon\right).$$

Since $\prod_{k=1}^{i+1} g_k$ divides f_j for all $j > i$, it follows that $f_i \in R$ with $\prod_{k=1}^{i} g_k$ dividing f_i. Repeating this process, we can define $f_i \in R$ for $0 \leqslant i \leqslant n$. Moreover, the equation $y^{-q_i+q_{i+1}}\,\mathrm{in}_{>_+}(f_i) = \mathrm{in}_{>_+}(g_{i+1}f_i) = x^{p_i-p_{i+1}}\,\mathrm{in}_{>_+}(f_{i+1})$ establishes that $\mathrm{in}_{>_+}(f_i) = x^{p_i}y^{q_i}$. With this notation, we define the ideal $I := \langle f_0, \ldots, f_n\rangle \subseteq K[x, y]$.

We next show that $\mathrm{in}_{>_+}(I \otimes_K k(\mathfrak{p})) = L_h \otimes_K k(\mathfrak{p})$ where $k(\mathfrak{p}) := K_\mathfrak{p}/\mathfrak{p}K_\mathfrak{p}$ is the residue field of the point $\mathfrak{p} \in \mathrm{Spec}(K)$. Since the minimal generators of L_h are the initial terms with respect to $>_+$ of the defining generators of $I(\mathfrak{p}) := I \otimes_K k(\mathfrak{p})$, it suffices to show that these generators form a Gröbner basis. By Buchberger's criterion [4, Exercise 15.19], we need only prove that the S-polynomials reduce to zero modulo the generators of $I(\mathfrak{p})$ for pairs of generators corresponding to the minimal syzygies of L_h. The minimal syzygies of a monomial ideal in $R(\mathfrak{p}) := R \otimes_K k(\mathfrak{p}) = k(\mathfrak{p})[x, y]$ are indexed by adjacent pairs of minimal generators; see [22, Proposition 3.1]. The S-polynomial for the adjacent generators f_{i-1}, f_i is

$$\begin{aligned}
&y^{-q_{i-1}+q_i} f_{i-1} - x^{p_{i-1}-p_i} f_i \\
&\quad = \left(y^{-q_{i-1}+q_i} - g_i\right) f_{i-1} + \sum_{(i-1,u,v)\in T_{\leqslant 0}(L_h)} c_{u,v}^{i-1} x^{u-p_\varepsilon} y^{v-q_{i-1}+q_i-q_\varepsilon} f_\varepsilon \\
&\quad = \sum_{(i-1,u,v)\in T_{\leqslant 0}(L_h)} c_{u,v}^{i-1} x^{u-p_\varepsilon} y^{v-q_{i-1}+q_i-q_\varepsilon} f_\varepsilon - \left(\sum_{(i,p_i,v)\in T(L_h)} c_{p_i,v}^{i} y^{v-q_{i-1}} \right) f_{i-1}.
\end{aligned}$$

Since the initial terms of all the summands in the last expression are less than the monomial $y^{-q_{i-1}+q_i} \operatorname{in}_{>_+}(f_{i-1})$, we conclude that this S-polynomial reduces to zero modulo the generators of $I(\mathfrak{p})$. It follows from [4, Theorem 15.3] that $h : A \to \mathbb{N}$ is the Hilbert function of $R(\mathfrak{p})/I(\mathfrak{p})$.

We now use this to show that I is admissible. Since $\dim_{k(\mathfrak{p})}(R(\mathfrak{p})/I(\mathfrak{p}))_a = h(a)$ for all $\mathfrak{p} \in \operatorname{Spec}(K)$, Nakayama's Lemma implies that the $K_\mathfrak{p}$-module $(R_\mathfrak{p}/I_\mathfrak{p})_a$ requires at most $h(a)$ generators. However, the rank of the $K_\mathfrak{p}$-module $(R_\mathfrak{p}/I_\mathfrak{p})_a$ is also bounded above by $\dim_{k(0)}(R(0)/I(0))_a$ and the Hilbert function at the generic point $\langle 0\rangle \in \operatorname{Spec}(K)$ also equals $h(a)$. Hence, $(R/I)_a$ is a locally free K-module of constant rank $h(a)$ on $\operatorname{Spec}(K)$. The map $\tau : \mathbb{A}^d \to \operatorname{Hilb}_S^h$ determined by the admissible R-ideal I is injective, so the dimension of the image is d. $\square$

We conclude with the proof of the main result.

Proof of Theorem 1.1. We first show that Hilb_S^h is nonsingular when $S = \Bbbk[x, y]$ and $\Bbbk$ is a field. An ideal in $S = \Bbbk[x, y]$ with codimension greater than one has finite colength. Hence we may assume, by Theorem 2.5, that $|h| := \sum_{a\in A} h(a) < \infty$. Given a closed point on Hilb_S^h, Lemma 3.9 shows that the Gröbner degenerations of the corresponding ideal I' give an irreducible rational curve on Hilb_S^h that contains the points corresponding to I' and $\operatorname{in}_{>_-}(I')$. Since the dimension of the tangent space is upper semicontinuous, it suffices to demonstrate that each point on Hilb_S^h corresponding to a monomial ideal is nonsingular. Theorem 3.15 establishes that the points on Hilb_S^h corresponding to monomial ideals lie on a curve C in which the irreducible components are associated to positive significant arrows. It follows from Theorem 4.9 that the dimension of the tangent space is weakly increasing as we move along C from a point corresponding to a monomial ideal to the point corresponding to L_h. Proposition 5.1 proves that the point corresponding to L_h is nonsingular. We conclude that dimension of the tangent space is constant along C and Hilb_S^h is nonsingular. Theorem 3.15 also establishes that Hilb_S^h is connected, so it follows that Hilb_S^h is irreducible.

To complete the proof, let $S = \mathbb{Z}[x, y]$ and let $\eta : \operatorname{Hilb}_S^h \to \operatorname{Spec}(\mathbb{Z})$ be the canonical map. To show η is smooth, it suffices by [10, Theorem 17.5.1] to demonstrate that η is flat and, for each $\mathfrak{p} \in \operatorname{Spec}(\mathbb{Z})$, that the fiber $\eta^{-1}(\mathfrak{p})$ is smooth over the perfect field $\mathbb{Z}_\mathfrak{p}/\mathfrak{p}\mathbb{Z}_\mathfrak{p}$. Since each fiber $\eta^{-1}(\mathfrak{p})$ is $\operatorname{Hilb}_{\mathbb{Z}/\mathfrak{p}[x,y]}^h$ (for example, see [12, Lemma 3.14]), combining the first paragraph with [10, Corollaire 17.15.2] shows that each fiber is smooth. The lex-most ideal on each fiber produces a section of η, which implies that η is surjective. The image of this section is irreducible, since $\mathrm{Spec}(\mathbb{Z})$ is, and the first paragraph also shows the fibers $\eta^{-1}(\mathfrak{p})$ are all irreducible as well, so it follows that Hilb_S^h is irreducible. Hence, the underlying reduced scheme $(\mathrm{Hilb}_S^h)_{\mathrm{red}}$ is irreducible and dominates $\mathrm{Spec}(\mathbb{Z})$, so [14, Proposition III.9.7] establishes that the canonical map $(\mathrm{Hilb}_S^h)_{\mathrm{red}} \to \mathrm{Spec}(\mathbb{Z})$ is flat. The fact that the nilradical of Hilb_S^h is the zero sheaf, so $\mathrm{Hilb}_S^h = (\mathrm{Hilb}_S^h)_{\mathrm{red}}$, can then be deduced from the fact that each fiber $\eta^{-1}(\mathfrak{p})$ is reduced. Therefore, we conclude that Hilb_S^h is smooth and irreducible over $\mathbb{Z}$. $\square$

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