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$\mathscr{D}$-Modules on Smooth Toric Varieties

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Let X be a smooth toric variety. Cox introduced the homogeneous coordinate ring S of X and its irrelevant ideal $\mathfrak{b}$. Let A denote the ring of differential operators on Spec(S). We show that the category of $\mathscr{D}$-modules on X is equivalent to a subcategory of graded A-modules modulo $\mathfrak{b}$-torsion. Additionally, we prove that the characteristic variety of a $\mathscr{D}$-module is a geometric quotient of an open subset of the characteristic variety of the associated A-module and that holonomic $\mathscr{D}$-modules correspond to holonomic A-modules.

1. INTRODUCTION

Let X be a smooth toric variety over a field k. Cox [2] introduced the homogeneous coordinate ring S of X and the irrelevant ideal $\mathfrak{b}$. The k-algebra S is a polynomial ring with one variable for each one-dimensional cone in the fan Δ defining X and a natural grading by the class group Cl(X). The monomial ideal $\mathfrak{b} \subset S$ encodes the combinatorial structure

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of Δ. The following theorem of Cox [2] indicates the significance of the pair $(S, \mathfrak{b})$. We write $\mathscr{O}$-Mod for the category of quasi-coherent sheaves on X and S-GrMod for the category of graded S-modules. A graded S-module F is called $\mathfrak{b}$-torsion if, for all $f \in F$, there exists $\ell > 0$ such that $\mathfrak{b}^\ell f = 0$. Let $\mathfrak{b}$-Tors denote the full subcategory of $\mathfrak{b}$-torsion modules.

THEOREM (Cox). (1) *The category* $\mathscr{O}$-Mod *is equivalent to the quotient category* S-$\mathsf{GrMod}/\mathfrak{b}$-$\mathsf{Tors}$.

(2) *The variety* X *is a geometric quotient of* $\operatorname{Spec}(S)\backslash\operatorname{Var}(\mathfrak{b})$ *by a suitable torus action*.

When $X = \mathbb{P}^n$, this is Serre's description of quasi-coherent sheaves on projective space and the classical construction of projective space.

The aim of this paper is to provide the $\mathscr{D}$-module version of this theorem; $\mathscr{D}$ denotes the sheaf of differential operators on X. To state the analogue of the first part, we introduce the following notation. We write $\mathscr{D}$-Mod for the category of left $\mathscr{D}$-modules on X. The ring of differential operators on $\operatorname{Spec}(S)$ is the Weyl algebra A; it also has a natural $\operatorname{Cl}(X)$-grading. To each element $\bar{\mathbf{u}}$ in $\operatorname{Cl}(X)^\vee = \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Cl}(X), \mathbb{Z})$ we associate an "Euler" operator $\theta_{\bar{\mathbf{u}}} \in A$ [see (2) for the precise definition]. The full subcategory of graded left A-modules F satisfying $(\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle) \cdot F_{\bar{\mathbf{a}}} = 0$ for all $\bar{\mathbf{a}} \in \operatorname{Cl}(X)$ and all $\bar{\mathbf{u}} \in \operatorname{Cl}(X)^\vee$ is denoted A-GrMod_θ.

THEOREM 1.1. *The quotient category* A-$\mathsf{GrMod}_\theta/\mathfrak{b}$-$\mathsf{Tors}$ *is equivalent to the category* $\mathscr{D}$-Mod.

The special case, when X is a projective space, can be found in Section VII.9.2 of Borel [1].

This categorical equivalence is given by two functors. The first takes an object F in A-GrMod_θ to the $\mathscr{D}$-module $\widetilde{F}$ whose sections over the affine open subset U_σ associated to $\sigma \in \Delta$ are $(F_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$. The second maps a $\mathscr{D}$-module $\mathscr{F}$ to $\Gamma_L(\mathscr{F}) = \bigoplus_{\bar{\mathbf{a}} \in \operatorname{Cl}(X)} H^0(X, \mathscr{O}(\bar{\mathbf{a}}) \otimes \mathscr{F})$. In fact, if F is finitely generated then $\widetilde{F}$ is coherent and if $\mathscr{F}$ is coherent then it is of the form $\widetilde{F}$ for some finitely generated graded A-module F. Our analysis of the Γ_L extends the work of Musson [11] and Jones [7] on rings of twisted differential operators on toric varieties.

Our second major result is

THEOREM 1.2. *If* $F \in A$-GrMod_θ *is finitely generated, then the characteristic variety of* $\widetilde{F}$ *is a geometric quotient of a suitable open subset of the characteristic variety of* F.

Moreover, given a finitely generated $F \in A$-GrMod_θ which has no $\mathfrak{b}$-torsion, we show that the dimension of $\widetilde{F}$ is equal to the dimension of F

minus the rank of $\mathrm{Cl}(X)$. In particular, holonomic A-modules correspond to holonomic $\mathscr{D}$-modules.

The category of modules over the Weyl algebra is a well-studied algebraic object and we hope to study $\mathscr{D}$-modules on X by using these methods. In particular, effective algorithms have been developed for $\mathscr{D}$-modules on affine space; for example, see the work of Oaku [12], Walther [16], Saito *et al.* [15], and Oaku and Takayama [13]. It would be interesting to use our results to extend these methods to smooth toric varieties.

We expect that many of our results, in particular Theorem 3.4 and Theorem 4.2, are valid for a simplicial toric variety if one replaces S and A with the subrings $\bigoplus_{\bar{\mathbf{b}}\in\mathrm{Pic}(X)} S_{\bar{\mathbf{b}}}$ and $\bigoplus_{\bar{\mathbf{b}}\in\mathrm{Pic}(X)} A_{\bar{\mathbf{b}}}$. Recall that, for a simplicial toric variety, Cox [2] shows that quasi-coherent sheaves correspond to graded modules over this Picard group graded subring of S modulo torsion. For simplicity, we present the smooth case and leave the possible generalizations to the reader.

The contents of this paper are as follows: The second section reviews the basics about toric varieties, the Weyl algebra, and $\mathscr{D}$-modules. In the third section, we determine the A-module associated with the sheaf $\mathscr{D}\otimes\mathscr{O}(\bar{\mathbf{b}})$. We introduce the $\mathrm{Cl}(X)\times\mathrm{Cl}(X)$-graded A-A bimodule

$$D=\bigoplus_{\bar{\mathbf{b}}\in\mathrm{Cl}(X)}\frac{A(\bar{\mathbf{b}})}{A\cdot(\theta_{\bar{\mathbf{u}}}+\langle\bar{\mathbf{u}},\bar{\mathbf{b}}\rangle:\bar{\mathbf{u}}\in\mathrm{Cl}(X)^{\vee})}$$

and construct a morphism $\eta\colon D\to\bigoplus_{(\bar{\mathbf{a}},\bar{\mathbf{b}})\in\mathrm{Cl}(X)^2}H^0(X,\mathscr{O}(\bar{\mathbf{a}})\otimes\mathscr{D}\otimes\mathscr{O}(\bar{\mathbf{b}}))$. We prove that η is an isomorphism in two steps. We first show that η induces an isomorphism on the associated sheaves. We then establish that $H^0_{\mathfrak{b}}(D)=H^1_{\mathfrak{b}}(D)=0$. The first step is a local statement and can be reduced to results of Musson [10]. We provide a direct proof using simplifications due to Jones [6]. The fourth section contains the proof of Theorem 1.1. We establish this result for both left and right $\mathscr{D}$-modules. In general, there is an equivalence between these and we show how this is induced at the level of A-modules. In the last section, we prove Theorem 1.2 and related dimension results.

2. BACKGROUND

We collect here a number of standard definitions, results, and notation. Throughout this paper, we work over an algebraically closed field k of characteristic zero.

Toric Varieties. Let X be a smooth toric variety determined by the fan Δ in $N\cong\mathbb{Z}^n$. We write $\mathbf{v}_1,\dots,\mathbf{v}_d$ for the unique lattice vectors generating the one-dimensional cones in Δ and we assume that the $\mathbf{v}_i$ span

$N \otimes_{\mathbb{Z}} \mathbb{R}$. Each $\mathbf{v}_i$ corresponds to an irreducible torus invariant Weil divisor in X. Since these divisors generate the torus invariant Weil divisors, we may identify the group of torus invariant Weil divisors with $\mathbb{Z}^d$. Let $\mathbf{e}_i$ denote the standard basis for $\mathbb{Z}^d$ and set $\mathbf{e} = \mathbf{e}_1 + \cdots + \mathbf{e}_d$.

There is a short exact sequence

$$0 \longrightarrow N^\vee \xrightarrow{\iota} \mathbb{Z}^d \xrightarrow{\overline{()}} \mathrm{Cl}(X) \longrightarrow 0, \tag{1}$$

where $\iota(\mathbf{p}) = \langle \mathbf{p}, \mathbf{v}_1 \rangle \mathbf{e}_1 + \cdots + \langle \mathbf{p}, \mathbf{v}_d \rangle \mathbf{e}_d$ and the second map $\mathbf{a} \mapsto \bar{\mathbf{a}}$ is the projection from Weil divisors to the divisor class group. Since X is smooth, the divisor class group $\mathrm{Cl}(X)$ is isomorphic to the Picard group $\mathrm{Pic}(X)$. In particular, the invertible sheaf (line bundle) associated to $\bar{\mathbf{a}} \in \mathrm{Cl}(X)$ is denoted $\mathscr{O}(\bar{\mathbf{a}})$.

Following Cox [2], the homogeneous coordinate ring of X is the polynomial ring $S = k[x_1, \ldots, x_d]$ with a $\mathrm{Cl}(X)$-grading induced by

$$\deg(x^{\mathbf{a}}) = \deg(x_1^{\mathbf{a}_1} \cdots x_d^{\mathbf{a}_d}) = \bar{\mathbf{a}} \in \mathrm{Cl}(X).$$

For a cone $\sigma \in \Delta$, $\hat{\sigma}$ is the set $\{i : \mathbf{v}_i \notin \sigma\}$ and $x^{\hat{\sigma}} = \Pi_{\mathbf{v}_i \notin \sigma} x_i$ is the associated monomial in S. The irrelevant ideal of X is the reduced monomial ideal $\mathfrak{b} = (x^{\hat{\sigma}} \colon \sigma \in \Delta)$.

Recall that each cone $\sigma \in \Delta$ corresponds to an open affine subset of X, $U_\sigma \cong \mathrm{Spec}(S_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$. Every graded S-module F gives rise to a quasi-coherent sheaf on X, denoted by $\widetilde{F}$, which corresponds to the module $(F_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$ over U_σ. If F is finitely generated over S, then $\widetilde{F}$ is a coherent $\mathscr{O}$-module; $\mathscr{O}$ denotes the structure sheaf on X. Moreover, every quasi-coherent sheaf on X is of the form $\widetilde{F}$ for some graded S-module F, and if the sheaf is coherent then F can be taken to be finitely generated. For an S-module F, we have $\widetilde{F} = 0$ if and only if $F = H^0_{\mathfrak{b}}(F)$; in other words, F is $\mathfrak{b}$-torsion.

Weyl Algebra. By definition, the dth Weyl algebra is

$$A = \frac{k\{x_1, \ldots x_d, \partial_1, \ldots, \partial_d\}}{\begin{pmatrix} x_i x_j - x_j x_i = 0 \\ \partial_i \partial_j - \partial_j \partial_i = 0 \\ \partial_i x_j - x_j \partial_i = \delta_{ij} \end{pmatrix}}$$

The canonical ring morphism $S \hookrightarrow A$ provides A with the structure of a left S-module. As in the case of S, A has a $\mathrm{Cl}(X)$-grading given by

$$\deg(x^{\mathbf{a}} \partial^{\mathbf{b}}) = \deg(x_1^{\mathbf{a}_1} \cdots x_d^{\mathbf{a}_d} \partial_1^{\mathbf{b}_1} \cdots \partial_d^{\mathbf{b}_d}) = \bar{\mathbf{a}} - \bar{\mathbf{b}} \in \mathrm{Cl}(X),$$

and the $\bar{\mathbf{a}}$th graded component of A is denoted $A_{\bar{\mathbf{a}}}$. For each element $\bar{\mathbf{u}}$ of $\mathrm{Cl}(X)^\vee = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(X), \mathbb{Z})$ we have an Euler operator

$$\theta_{\bar{\mathbf{u}}} = \langle \bar{\mathbf{u}}, \bar{\mathbf{e}}_1 \rangle \theta_1 + \cdots + \langle \bar{\mathbf{u}}, \bar{\mathbf{e}}_d \rangle \theta_d, \tag{2}$$

where $\theta_i = x_i \partial_i$. Note that $\theta_{\bar{\mathbf{u}}}$ has degree zero.

The Weyl algebra A is isomorphic to the ring of differential operators on $\mathbb{A}^d$. The natural action of A on a polynomial $f \in S$ is $x_i \bullet f = x_i \cdot f$ and $\partial_i \bullet f = \partial f / \partial x_i$. Since S is also a subring of A, the symbol $\bullet$ helps to distinguish this action from the product $\cdot$: $A \times A \to A$. The ring of differential operators on $S_{x^{\mathbf{a}}} = S[x^{-\mathbf{a}}]$ is denoted $A_{x^{\mathbf{a}}}$ and is equal to the localization $A[x^{-\mathbf{a}}]$.

$\mathcal{D}$-Modules. The sheaf of (algebraic) differential operators on X is denoted $\mathcal{D}$. On an affine open subset $U \subseteq X$, $H^0(U, \mathcal{D}) = \bigcup_{i \geq 0} \mathcal{D}^i(U)$ where $\mathcal{D}^0(U) = H^0(U, \mathcal{O})$ and

$$\mathcal{D}^i(U) := \left\{ s \in \mathrm{End}_k(H^0(U, \mathcal{O})) : \begin{array}{l} fs - sf \in \mathcal{D}^{i-1}(U) \text{ for} \\ \text{all } f \in H^0(U, \mathcal{O}) \end{array} \right\}.$$

A $\mathcal{D}$-module is a sheaf $\mathcal{F}$ on X which is quasi-coherent as an $\mathcal{O}$-module and has the structure of a module over $\mathcal{D}$. A $\mathcal{D}$-module is coherent if it is locally finitely generated over $\mathcal{D}$. We write $\mathcal{D}$-Mod and Mod-$\mathcal{D}$ for the categories of left and right $\mathcal{D}$-modules, respectively. The full subcategories of coherent left and right $\mathcal{D}$-modules are denoted $\mathcal{D}$-Coh and Coh-$\mathcal{D}$.

3. SHEAVES OF DIFFERENTIAL OPERATORS

The goal of this section is to describe the left S-modules corresponding to twists of the sheaf of differential operators. Recall that, for a graded A-module F and $\bar{\mathbf{b}} \in \mathrm{Cl}(X)$, $F(\bar{\mathbf{b}})$ is the shift of F by $\bar{\mathbf{b}}$: $F(\bar{\mathbf{b}})_{\bar{\mathbf{a}}} = F_{\bar{\mathbf{b}}+\bar{\mathbf{a}}}$. We define the graded left A-modules as

$$D_L(\bar{\mathbf{b}}) = \frac{A(\bar{\mathbf{b}})}{A \cdot (\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle : \bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee)}$$

and $D = \bigoplus_{\bar{\mathbf{b}} \in \mathrm{Cl}(X)} D_L(\bar{\mathbf{b}})$. Note that D has a $\mathrm{Cl}(X) \times \mathrm{Cl}(X)$-grading where $D_{(\bar{\mathbf{a}}, \bar{\mathbf{b}})} = D_L(\bar{\mathbf{b}})_{\bar{\mathbf{a}}}$.

LEMMA 3.1. *The module D is a graded A–A bimodule.*

Proof. Multiplication in the ring A yields the right action of A on D,

$$\begin{array}{ccc} A(\bar{\mathbf{b}})_{\bar{\mathbf{a}}} \otimes_k A_{\bar{\mathbf{b}}'} & \longrightarrow & A(\bar{\mathbf{b}} + \bar{\mathbf{b}}')_{\bar{\mathbf{a}}} \\ \big\downarrow & & \big\downarrow \\ D_L(\bar{\mathbf{b}})_{\bar{\mathbf{a}}} \otimes_k A_{\bar{\mathbf{b}}'} & \longrightarrow & D_L(\bar{\mathbf{b}} + \bar{\mathbf{b}}')_{\bar{\mathbf{a}}}. \end{array}$$

To see that the induced map is well-defined, observe that, for all elements $f \in A_{\bar{\mathbf{b}}'}$ and $\bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee$, we have

$$(\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle) \cdot f = f \cdot (\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle) + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}}' \rangle \cdot f = f \cdot (\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} + \bar{\mathbf{b}}' \rangle).$$

This action is clearly compatible with the left A-module structure. It follows that D is an A–A bimodule. Since $f \in A_{\bar{\mathbf{a}}'}$ and $g \in A_{\bar{\mathbf{b}}'}$ imply $f \cdot D_{(\bar{\mathbf{a}}, \bar{\mathbf{b}})} \cdot g \subseteq D_{(\bar{\mathbf{a}}+\bar{\mathbf{a}}', \bar{\mathbf{b}}+\bar{\mathbf{b}}')}$, D is bigraded. In other words, if we let A° denote the opposite algebra, then $A \otimes_k A^\circ$ is a $\mathrm{Cl}(X)^2$-graded ring and D is a graded module over $A \otimes_k A^\circ$. ∎

Analogously, we define right A-modules,

$$D_R(\bar{\mathbf{a}}) = \frac{A(\bar{\mathbf{a}})}{(\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle : \bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee) \cdot A}$$

and $D' = \bigoplus_{\bar{\mathbf{a}} \in \mathrm{Cl}(X)} D_R(\bar{\mathbf{a}})$. Again, D' is a $\mathrm{Cl}(X) \times \mathrm{Cl}(X)$-graded A–A bimodule where the multiplication on the left is induced by the multiplication in the Weyl algebra. In fact, we obtain the same module.

LEMMA 3.2. *There is a canonical identification $D = D'$ which respects the graded bimodule structure.*

Proof. Since we have

$$D_{(\bar{\mathbf{a}}, \bar{\mathbf{b}})} = \frac{A_{\bar{\mathbf{a}}+\bar{\mathbf{b}}}}{A_{\bar{\mathbf{a}}+\bar{\mathbf{b}}} \cdot (\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle : \bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee)} \quad \text{and}$$

$$D'_{(\bar{\mathbf{a}}, \bar{\mathbf{b}})} = \frac{A_{\bar{\mathbf{a}}+\bar{\mathbf{b}}}}{(\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle : \bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee) \cdot A_{\bar{\mathbf{a}}+\bar{\mathbf{b}}}},$$

it is enough to show that $A_{\bar{\mathbf{a}}+\bar{\mathbf{b}}} \cdot (\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle) = (\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle) \cdot A_{\bar{\mathbf{a}}+\bar{\mathbf{b}}}$. However, for every $f \in A_{\bar{\mathbf{a}}+\bar{\mathbf{b}}}$ we have

$$f \cdot (\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle) = (\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle) \cdot f - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} + \bar{\mathbf{b}} \rangle \cdot f = (\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle) \cdot f,$$

which establishes the lemma. ∎

Now, the direct sum of sections of twists of $\mathscr{D}$ has an A–A bimodule structure.

LEMMA 3.3. *The direct sum*

$$\bigoplus_{(\bar{\mathbf{a}}, \bar{\mathbf{b}}) \in \mathrm{Cl}(X)^2} H^0(X, \mathscr{O}(\bar{\mathbf{a}}) \otimes \mathscr{D} \otimes \mathscr{O}(\bar{\mathbf{b}}))$$

is an A–A bimodule.

Proof. It suffices to give k-linear maps,

$$\mu\colon A_{\bar{\mathbf{a}}'} \otimes_k H^0(X, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}})) \otimes_k A_{\bar{\mathbf{b}}'} \longrightarrow H^0(X, \mathcal{O}(\bar{\mathbf{a}} + \bar{\mathbf{a}}') \otimes \mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}} + \bar{\mathbf{b}}')). \tag{3}$$

Locally, a section $s \in H^0(U_\sigma, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}}))$ can be identified with an element of $\mathrm{Hom}_k((S_{x^{\hat{\sigma}}})_{-\bar{\mathbf{b}}}, (S_{x^{\hat{\sigma}}})_{\bar{\mathbf{a}}})$, where σ is a cone in Δ. Moreover, the action of $f \in A_{\bar{\mathbf{a}}}$ on S descends to the action on $S_{x^{\hat{\sigma}}}$ which increases degrees by $\bar{\mathbf{a}}$. Thus, we may define $(\mu|_{U_\sigma})(f \otimes s \otimes g) = f \circ s \circ g$. One verifies that $\mu|U_\sigma$ maps into $H^0(U_\sigma, \mathcal{O}(\bar{\mathbf{a}} + \bar{\mathbf{a}}') \otimes \mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}} + \bar{\mathbf{b}}'))$ and that these local definitions glue together to give the required map. ∎

We next construct a morphism of graded A–A bimodules,

$$\eta\colon D \longrightarrow \bigoplus_{(\bar{\mathbf{a}}, \bar{\mathbf{b}})} H^0(X, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}})). \tag{4}$$

Since η is a graded k-linear morphism, it is enough to define $\eta(f)$ for $f \in D_{(\bar{\mathbf{a}}, \bar{\mathbf{b}})}$. For $f \in D_{(\bar{\mathbf{a}}, \bar{\mathbf{b}})}$, let $\eta(f)$ be the section of $H^0(X, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}}))$ whose restriction over each U_σ corresponds to the map induced by the action of $f : (S_{x^{\hat{\sigma}}})_{-\bar{\mathbf{b}}} \longrightarrow (S_{x^{\hat{\sigma}}})_{\bar{\mathbf{a}}}$. To see that $\eta(f)$ is well-defined, consider $f \in A_{\bar{\mathbf{a}}+\bar{\mathbf{b}}} \cdot (\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle)$. It follows that, for $g \in (S_{x^{\hat{\sigma}}})_{-\bar{\mathbf{b}}}$, we have

$$(\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle) \bullet g = -\langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle \cdot g + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle \cdot g = 0$$

and therefore $\eta(f) = 0$. It is clear that η is a morphism of graded A–A bimodules. The main result of this section is the following.

THEOREM 3.4. *The morphism η [see Eq. (4)] is an isomorphism of graded A–A bimodules.*

Before proving Theorem 3.4, we collect some local results. We first consider a local version of η. By composing η with the restriction to U_σ where $\sigma \in \Delta$, we obtain a morphism of left S-modules

$$D \longrightarrow \bigoplus_{(\bar{\mathbf{a}}, \bar{\mathbf{b}}) \in \mathrm{Cl}(X)^2} H^0(U_\sigma, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}})),$$

which induces a morphism

$$\eta^\sigma : D_{x^{\hat{\sigma}}} \longrightarrow \bigoplus_{(\bar{\mathbf{a}}, \bar{\mathbf{b}}) \in \mathrm{Cl}(X)^2} H^0(U_\sigma, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}})).$$

Taking the degree zero component yields the ring morphism

$$\bar{\varphi}_\sigma = \eta^\sigma_{(\bar{\mathbf{0}}, \bar{\mathbf{0}})}\colon \frac{(A_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}}{(A_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}} \cdot (\theta_{\bar{\mathbf{u}}} : \bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee)} \longrightarrow H^0(U_\sigma, \mathcal{D}). \tag{5}$$

Understanding $\bar{\varphi}_\sigma$ is an important ingredient in establishing Theorem 3.4. We begin by studying the map $\varphi_\sigma : (A_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}} \to H^0(U_\sigma, \mathscr{D})$ which induces $\bar{\varphi}_\sigma$.

By definition, $H^0(U_\sigma, \mathscr{D})$ is the ring of differential operators on the ring $(S_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$. However, the inclusion ι_σ: $\sigma^\vee \cap N^\vee \hookrightarrow \mathbb{Z}^d$ induces a ring isomorphism (denoted by the same name) ι_σ: $k[\sigma^\vee \cap N^\vee] \xrightarrow{\cong} (S_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$. To see this, observe that $x^{\iota(\mathbf{p})} \in S_{x^{\hat{\sigma}}}$ if and only if $\langle \mathbf{p}, \mathbf{v}_i \rangle \geq 0$, for all $\mathbf{v}_i \in \sigma$, which is equivalent to $\mathbf{p} \in \sigma^\vee$. It follows that the isomorphism ι_σ induces an isomorphism, called ψ_σ, from the differential operators on $(S_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$ to the differential operators R_σ on $k[\sigma^\vee \cap N^\vee]$. More explicitly, we have $\psi_\sigma(f) = \iota_\sigma^{-1} \circ f \circ \iota_\sigma$. We will actually focus on the morphism $\psi_\sigma \circ \varphi_\sigma : (A_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}} \to R_\sigma$.

Following Musson [10] and Jones [6], we decompose $(A_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$ and R_σ under the appropriate torus actions and express $\psi_\sigma \circ \varphi_\sigma$ in terms of these decompositions. To be more concrete, we identify $N^\vee$ with $\mathbb{Z}^n$ by fixing a basis. Let $\varepsilon_1, \dots, \varepsilon_n$ denote the standard basis of $\mathbb{Z}^n$ and let $\varepsilon = \varepsilon_1 + \cdots + \varepsilon_n$. We continue to call the natural embedding ι: $N^\vee = \mathbb{Z}^n \to \mathbb{Z}^d$.

The torus $(k^*)^d$ acts on $k[x_1^{\pm 1}, \dots, x_d^{\pm 1}]$ by $\lambda * (x^{\mathbf{a}}) = \lambda^{\mathbf{a}} x^{\mathbf{a}}$ which produces an action on A_{x^e} given by $\lambda * (x^{\mathbf{a}} \partial^{\mathbf{b}}) = \lambda^{\mathbf{a}-\mathbf{b}} x^{\mathbf{a}} \partial^{\mathbf{b}}$ for $\lambda \in (k^*)^d$. The corresponding eigenspace decomposition is $A_{x^e} = \bigoplus_{a \in \mathbb{Z}^d} x^{\mathbf{a}} \cdot W$, where W is the polynomial ring $k[\theta_1, \dots \theta_d]$. Taking the degree zero part, we have $(A_{x^e})_{\bar{\mathbf{0}}} = \bigoplus_{\mathbf{p} \in \mathbb{Z}^n} x^{\iota(\mathbf{p})} \cdot W$. Since $(A_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$ is invariant under the action of $(k^*)^d$, the decomposition of $(A_{x^e})_{\bar{\mathbf{0}}}$ yields $(A_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}} = \bigoplus_{\mathbf{p} \in \mathbb{Z}^n} x^{\iota(\mathbf{p})} \cdot J(\mathbf{p})$, where $J(\mathbf{p})$ is an ideal in W. To describe $J(\mathbf{p})$, recall that $A_{x^{\hat{\sigma}}}$ is the ring of differential operators on $S_{x^{\hat{\sigma}}}$. Thus, if $g \in W$, then $x^{\iota(\mathbf{p})} g$ belongs to $(A_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$ if and only if $(x^{\iota(\mathbf{p})} g) \bullet S_{x^{\hat{\sigma}}} \subseteq S_{x^{\hat{\sigma}}}$. Equivalently, for every $x^{\mathbf{a}}$ satisfying $\mathbf{a}_i \geq 0$ when $\mathbf{v}_i \in \sigma$, we have $g(\mathbf{a}) x^{\iota(\mathbf{p})+\mathbf{a}} \in S_{x^{\hat{\sigma}}}$. We conclude that $J(\mathbf{p})$ is the ideal of polynomials vanishing on

$$Z(\mathbf{p}) = \left\{ \mathbf{a} \in \mathbb{Z}^d \colon \begin{array}{l} \mathbf{a}_i \geq 0 \text{ if } \mathbf{v}_i \in \sigma \text{ and} \\ \iota(\mathbf{p})_j + \mathbf{a}_j < 0 \text{ for some } \mathbf{v}_j \in \sigma \end{array} \right\}.$$

Analogously, the affine space $\mathbb{A}_k^n$ has, as its associated Weyl algebra,

$$A' = \frac{k\{y_1, \dots y_n, \eth_1, \dots, \eth_n\}}{\begin{pmatrix} y_i y_j - y_j y_i = 0 \\ \eth_i \eth_j - \eth_j \eth_i = 0 \\ \eth_i y_j - y_j \eth_i = \delta_{ij} \end{pmatrix}}$$

and the torus $(k^*)^n$ acts on the Laurent ring $k[y_1^{\pm 1}, \dots, y_n^{\pm 1}] = k[N^\vee]$, inducing an action on A'_{y^e}. The corresponding eigenspace decomposition is $A'_{y^\epsilon} = \bigoplus_{\mathbf{p} \in \mathbb{Z}^n} y^{\mathbf{p}} \cdot W'$, where W' is the polynomial ring $k[\vartheta_1, \dots, \vartheta_n]$ and

$\vartheta_i = y_i \eth_i$. The inclusion $R_\sigma \hookrightarrow A'_{y^\epsilon}$ identifies R_σ with

$$\{f \in A'_{y^\epsilon} : f \bullet (k[\sigma^\vee \cap N^\vee]) \subseteq k[\sigma^\vee \cap N^\vee]\}.$$

Since this condition is torus invariant, R_σ is also torus invariant and we obtain $R_\sigma = \bigoplus_{\mathbf{p} \in \mathbb{Z}^n} y^{\mathbf{p}} \cdot I(\mathbf{p})$, where

$$I(\mathbf{p}) = \{f \in W' : (y^{\mathbf{p}} f) \bullet (k[\sigma^\vee \cap N^\vee]) \subseteq k[\sigma^\vee \cap N^\vee]\}.$$

Identifying W' with the coordinate ring of $\mathbb{A}^n$, we have $f \bullet y^{\mathbf{q}} = f(\mathbf{q}) y^{\mathbf{q}}$ for every $\mathbf{q} \in N^\vee$. Hence, $I(\mathbf{p})$ is the ideal of polynomials vanishing on

$$Y(\mathbf{p}) = \{\mathbf{q} \in \sigma^\vee \cap N^\vee : \mathbf{q} + \mathbf{p} \notin \sigma^\vee \cap N^\vee\}.$$

Finally, we define a map ρ: $W \longrightarrow W'$. The inclusion ι: $\mathbb{Z}^n \to \mathbb{Z}^d$ induces, by tensoring with k, a linear embedding (denoted by the same name) ι: $\mathbb{A}^n \to \mathbb{A}^d$ of the corresponding affine spaces. Identifying W and W' with the coordinate rings of $\mathbb{A}^d$ and $\mathbb{A}^n$ respectively, we obtain the ring homomorphism $\rho = \iota^*$: $W \to W'$. Clearly, ρ is surjective and $\mathrm{Ker}(\rho) = (\theta_{\bar{\mathbf{u}}} : \bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee)$.

With this notation, we have

LEMMA 3.5. *The ring homomorphism* $\psi_\sigma \circ \varphi_\sigma : (A_{x^\sigma})_{\bar{\mathbf{0}}} \longrightarrow R_\sigma$ *is given by* $x^{\iota(\mathbf{p})} \cdot g \mapsto y^{\mathbf{p}} \cdot \rho(g)$ *where* $\mathbf{p} \in \mathbb{Z}^n$ *and* $g \in J(\mathbf{p})$.

Proof. It suffices to show that $(\psi_\sigma \circ \varphi_\sigma)(x^{\iota(\mathbf{p})} g)$ and $y^{\mathbf{p}} \rho(g)$ have the same action on $y^{\mathbf{q}} \in k[\sigma^\vee \cap N^\vee]$. On one hand, we have

$$\begin{aligned}
(\psi_\sigma \circ \varphi_\sigma)(x^{\iota(\mathbf{p})} g) \bullet y^{\mathbf{q}} &= \iota^{-1}\big(\varphi_\sigma(x^{\iota(\mathbf{p})} g) \bullet \iota(y^{\mathbf{q}})\big) \\
&= \iota^{-1}\big(\varphi_\sigma(x^{\iota(\mathbf{p})} g) \bullet x^{\iota(\mathbf{q})}\big) \\
&= \iota^{-1}\big(g(\iota(\mathbf{q})) x^{\iota(\mathbf{p}+\mathbf{q})}\big) \\
&= g(\iota(\mathbf{q})) y^{\mathbf{p}+\mathbf{q}}.
\end{aligned}$$

On the other hand, we also have

$$(y^{\mathbf{p}} \rho(g)) \bullet y^{\mathbf{q}} = (\rho(g)(\mathbf{q}) y^{\mathbf{p}+\mathbf{q}}) = g(\iota(\mathbf{q})) y^{\mathbf{p}+\mathbf{q}},$$

which establishes the claim. ∎

Before returning our attention to $\bar{\varphi}_\sigma$, we need one more lemma.

LEMMA 3.6. *If the elements* θ_i *and* θ_j *in* $W = k[\theta_1, \ldots, \theta_d]$ *are distinct and correspond to the rays* $\mathbf{v}_i$ *and* $\mathbf{v}_j$ *in the same cone* σ, *then, for every pair of integers* m_i *and* m_j, *the elements* $\rho(\theta_i) + m_i$ *and* $\rho(\theta_j) + m_j$ *are linearly independent over the field* k.

Proof. Suppose otherwise, then we have $\rho(\theta_i) + m_i = c \cdot (\rho(\theta_j) + m_j)$ for some $c \in k$. It follows that $\rho(\theta_i) = c \cdot \rho(\theta_j)$ and hence

$$\sum_{\ell=1}^{n} \langle \varepsilon_\ell, \mathbf{v}_i \rangle \vartheta_\ell = c \cdot \left(\sum_{\ell=1}^{n} \langle \varepsilon_\ell, \mathbf{v}_j \rangle \vartheta_\ell \right).$$

We deduce that $\langle \varepsilon_\ell, \mathbf{v}_i \rangle = \langle \varepsilon_\ell, c \cdot \mathbf{v}_j \rangle$ for all ℓ and thus $\mathbf{v}_i = c \cdot \mathbf{v}_j$. However, the two rays in a strongly convex cone are linearly independent. ∎

We are now in a position to understand $\bar{\varphi}_\sigma$. In particular, we obtain the following proposition which is a special case of Musson's results on rings of differential operators; see Musson [10].

PROPOSITION 3.7 (Musson). *For every $\sigma \in \Delta$, the map $\bar{\varphi}_\sigma$ [see Eq. (5)] is an isomorphism of rings.*

Proof. The fact that $\bar{\varphi}_\sigma$ is a ring homomorphism follows directly from the definition. Thus, the assertion reduces to showing that $\psi_\sigma \circ \varphi_\sigma$ is surjective and describing its kernel. To achieve this, we determine the Zariski closures of $Y(\mathbf{p})$ and $Z(\mathbf{p})$. Since the Zariski closure of the set of integer points inside a rational polyhedral cone is the linear space spanned by that cone, it is easy to check that

$$\begin{aligned} \overline{Z(\mathbf{p})} &= \bigcup_{(i,\, m) \in \Lambda'} \{\mathbf{b} \in k^d : b_i = m\}, \\ \overline{Y(\mathbf{p})} &= \bigcup_{(i,\, m) \in \Lambda'} \{\mathbf{q} \in k^n : \iota(\mathbf{q})_i = m\} = \iota^{-1}(\overline{Z(\mathbf{p})}), \end{aligned} \tag{6}$$

where $\Lambda' = \{(i, m) : \mathbf{v}_i \in \sigma \text{ and } 0 \le m \le -\iota(\mathbf{p})_i\}$. We claim that $\rho(J(\mathbf{p})) = I(\mathbf{p})$. Indeed, Eq. (6) implies that $I(\mathbf{p}) = \sqrt{\rho(J(\mathbf{p}))}$, so it is enough to check that $\rho(J(\mathbf{p}))$ is a radical ideal. However, $J(\mathbf{p})$ is the principal ideal generated by

$$h_{\mathbf{p}} := \prod_{(i,m) \in \Lambda'} (\theta_i - m). \tag{7}$$

From the equation (7) and Lemma 3.6, we see that $\rho(J(\mathbf{p}))$ is reduced and hence $I(\mathbf{p}) = \rho(J(\mathbf{p}))$. Applying Lemma 3.5, it follows that $\psi_\sigma \circ \varphi_\sigma$ and $\bar{\varphi}_\sigma$ are surjective. To prove that $\psi_\sigma \circ \bar{\varphi}_\sigma$ is injective, recall that

$$\operatorname{Ker}(\rho) = (\theta_{\bar{\mathbf{u}}} : \bar{\mathbf{u}} \in \operatorname{Cl}(X)^\vee).$$

Thus, it is enough to show that we have $J(\mathbf{p}) \cap \operatorname{Ker}(\rho) = J(\mathbf{p}) \cdot \operatorname{Ker}(\rho)$, for every $\mathbf{p} \in \mathbb{Z}^n$. To see this, observe that $f \in J(\mathbf{p}) \cap \operatorname{Ker}(\rho)$ implies $f = h_{\mathbf{p}} f_1$. Therefore, it suffices to note that $\rho(h_{\mathbf{p}}) \neq 0$, which follows from Lemma 3.6. ∎

We now prove the main result in this section.

Proof of Theorem 3.4. We must show that, for every $\bar{\mathbf{b}} \in \mathrm{Cl}(X)$, the homomorphism of graded left A-modules

$$\eta_{(*,\bar{\mathbf{b}})}\colon D_L(\bar{\mathbf{b}}) \longrightarrow \bigoplus_{\bar{\mathbf{a}}\in\mathrm{Cl}(X)} H^0(X, \mathscr{O}(\bar{\mathbf{a}}) \otimes \mathscr{D} \otimes \mathscr{O}(\bar{\mathbf{b}}))$$

is an isomorphism.

The first step is to prove that $\eta_{(*,\bar{\mathbf{b}})}$ induces an isomorphism of the associated sheaves. Fix $\sigma \in \Delta$ and choose $\mathbf{b} \in \mathbb{Z}^d$, mapping to $\bar{\mathbf{b}}$ in $\mathrm{Cl}(X)$, such that $x^{\mathbf{b}}$ is an invertible element in $S_{x^{\hat{\sigma}}}$. Since the restriction of $\mathscr{O}(\bar{\mathbf{b}})$ to U_σ is trivial, one may always find such a $\mathbf{b}$. We then have a commutative diagram,

$$\begin{array}{ccc}
\dfrac{(A_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}}{(A_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}\cdot(\theta_{\bar{\mathbf{u}}} : \bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee)} & \xrightarrow{\bar{\varphi}_\sigma} & H^0(U_\sigma, \mathscr{D}) \\
\Big\downarrow {\scriptstyle \cdot x^{\mathbf{b}}} & & \Big\downarrow {\scriptstyle \otimes x^{\mathbf{b}}} \\
\dfrac{(A_{x^{\hat{\sigma}}})_{\bar{\mathbf{b}}}}{(A_{x^{\hat{\sigma}}})_{\bar{\mathbf{b}}}\cdot(\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}}\rangle : \bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee)} & \xrightarrow{\bar{\varphi}'_\sigma} & H^0(U_\sigma, \mathscr{D} \otimes \mathscr{O}(\bar{\mathbf{b}})),
\end{array}$$

where $\bar{\varphi}_\sigma$ is the morphism in Proposition 3.7 and $\bar{\varphi}'_\sigma$ is the analogous morphism induced by $\eta_{(\bar{\mathbf{0}},\bar{\mathbf{b}})}$. Now, Proposition 3.7 implies that $\bar{\varphi}_\sigma$ is an isomorphism and the vertical arrows are clearly isomorphisms. It follows that $\bar{\varphi}'_\sigma$ is an isomorphism and therefore $\eta_{(*,\bar{\mathbf{b}})}$ induces an isomorphism of the associated sheaves.

If F is a graded S-module, we write $\Gamma_L(\widetilde{F}) = \bigoplus_{\bar{\mathbf{a}}\in\mathrm{Cl}(X)} H^0(X, \mathscr{O}(\bar{\mathbf{a}}) \otimes \widetilde{F})$. For every such F, there is an exact sequence

$$0 \longrightarrow H^0_{\mathfrak{b}}(F) \longrightarrow F \longrightarrow \Gamma_L(\widetilde{F}) \longrightarrow H^1_{\mathfrak{b}}(F) \longrightarrow 0, \tag{8}$$

where $\mathfrak{b}$ is the irrelevant ideal; see Eisenbud *et al.* [3]. Hence, if $H^0_{\mathfrak{b}}(D_L(\bar{\mathbf{b}}))$ and $H^1_{\mathfrak{b}}(D_L(\bar{\mathbf{b}}))$ both vanish, then $\eta_{(*,\bar{\mathbf{b}})}$ is an isomorphism. We relegated these vanishing results to Propositions 3.8 and 3.9 below. ∎

Our first vanishing result is

PROPOSITION 3.8. *The element $x^{\mathbf{e}} \in \mathfrak{b}$ is not a zero divisor on $D_L(\bar{\mathbf{b}})$, and $H^0_{\mathfrak{b}}(D_L(\bar{\mathbf{b}})) = 0$.*

Proof. The first assertion implies the second, so it suffices to show that $x^{\mathbf{e}}$ is not a zero divisor. Every $\mathbf{a} \in \mathbb{Z}^d$ can be written uniquely as $\mathbf{a} = \mathbf{a}^+ - \mathbf{a}^-$, where $\mathbf{a}^+$ and $\mathbf{a}^-$ are non-negative and have disjoint support. Consider the action of the torus $(k^*)^d$ on the Weyl algebra A; the corresponding eigenspace decomposition is $A = \bigoplus_{\mathbf{a}\in\mathbb{Z}^d} x^{\mathbf{a}^+}\partial^{\mathbf{a}^-} \cdot W$, where $W = k[\theta_1, \ldots, \theta_d]$. Let $L_0 \subset W$ be the ideal generated by $\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}}\rangle$ for all $\bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee$. Since L_0 is generated by linear forms, it is a prime ideal.

Let L denote the left A-ideal $(\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle : \bar{\mathbf{u}} \in \mathrm{Cl}(X)^{\vee})$. With this notation, we have $L = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} x^{\mathbf{a}^+} \partial^{\mathbf{a}^-} \cdot L_0$.

For $x^{\mathbf{e}}$ to be a nonzero divisor on $D_L(\bar{\mathbf{b}})$, it suffices to prove that, for $g \in W$ and $\mathbf{a} \in \mathbb{Z}^d$, the relation $x^{\mathbf{e}} x^{\mathbf{a}^+} \partial^{a^-} g \in L$ implies $g \in L_0$. To accomplish this, we first note that

$$\begin{aligned} x^{\mathbf{e}} \cdot x^{\mathbf{a}^+} \partial^{\mathbf{a}^-} &= \left(\prod_{\mathbf{a}_i \geq 0} x_i^{\mathbf{a}_i + 1} \right) \left(\prod_{\mathbf{a}_i < 0} x_i \partial_i^{-\mathbf{a}_i} \right) \\ &= \left(\prod_{\mathbf{a}_i \geq 0} x_i^{\mathbf{a}_i + 1} \right) \left(\prod_{\mathbf{a}_i < 0} \partial_i^{-\mathbf{a}_i - 1} (\theta_i + \mathbf{a}_i + 1) \right) \\ &= x^{(\mathbf{a}+\mathbf{e})^+} \partial^{(\mathbf{a}+\mathbf{e})^-} \prod_{\mathbf{a}_i < 0} (\theta_i + \mathbf{a}_i + 1). \end{aligned}$$

Hence, $x^{\mathbf{e}} \cdot x^{\mathbf{a}^+} \partial^{\mathbf{a}^-} g \in L$ implies that $(\prod_{\mathbf{a}_i < 0} (\theta_i + \mathbf{a}_i + 1)) \cdot g \in L_0$. Suppose $g \notin L_0$. Since L_0 is a prime ideal, there exists an index i such that $\mathbf{a}_i < 0$ and $\theta_i + \mathbf{a}_i + 1 \in L_0$. Expressing $\theta_i + \mathbf{a}_i + 1$ in terms of the linear generators of L_0, it follows that there is $\bar{\mathbf{u}} \in \mathrm{Cl}(X)^{\vee} \otimes_{\mathbb{Z}} k$ such that $\theta_i = \theta_{\bar{\mathbf{u}}}$. However, for all $\mathbf{w} \in N^{\vee}$, the short exact sequence (1) implies $\langle \bar{\mathbf{u}}, \langle \mathbf{w}, \mathbf{v}_1 \rangle \bar{\mathbf{e}}_1 + \cdots + \langle \mathbf{w}, \mathbf{v}_d \rangle \bar{\mathbf{e}}_d \rangle = 0$, from which we deduce that $\langle \mathbf{w}, \mathbf{v}_i \rangle = 0$ and $\mathbf{v}_i = 0$, giving a contradiction. ∎

We end this section with

PROPOSITION 3.9. *The local cohomology module $H^1_{\mathfrak{b}}(D_L(\bar{\mathbf{b}}))$ vanishes.*

Proof. Since $x^{\mathbf{e}}$ is not a zero divisor on $D_L(\bar{\mathbf{b}})$, there is a short exact sequence of S-modules

$$0 \longrightarrow D_L(\bar{\mathbf{b}}) \xrightarrow{x^{\mathbf{e}} \cdot} D_L(\bar{\mathbf{b}}) \longrightarrow Q = \frac{D_L(\bar{\mathbf{b}})}{x^{\mathbf{e}} \cdot D_L(\bar{\mathbf{b}})} \longrightarrow 0,$$

and the long exact sequence of local cohomology gives

$$0 \longrightarrow H^0_{\mathfrak{b}}(Q) \longrightarrow H^1_{\mathfrak{b}}(D_L(\bar{\mathbf{b}})) \xrightarrow{x^{\mathbf{e}} \cdot} H^1_{\mathfrak{b}}(D_L(\bar{\mathbf{b}})). \tag{9}$$

Because every element in $H^1_{\mathfrak{b}}(D_L(\bar{\mathbf{b}}))$ is annihilated by a power of $\mathfrak{b}$, the injectivity of $x^{\mathbf{e}} \cdot$ in the exact sequence (9) would imply that $H^1_{\mathfrak{b}}(D_L(\bar{\mathbf{b}})) = 0$. Thus, it suffices to prove that $H^0_{\mathfrak{b}}(Q) = 0$.

Let K be the left A-ideal satisfying $Q = A(\bar{\mathbf{b}})/K$. To prove that $H^0_{\mathfrak{b}}(Q) = 0$ we must show that if $f \in A$ satisfies $(x^{\hat{\sigma}})^m f \in K$ for some $m \geq 1$ and all $\sigma \in \Delta$, then $f \in K$. Using the notation from the proof of Proposition 3.8, we have $K = L + x^{\mathbf{e}} \cdot A$. From the decomposition of A, we have

$$\begin{aligned} x^{\mathbf{e}} \cdot A &= \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} x^{\mathbf{e}} \cdot x^{\mathbf{a}^+} \partial^{\mathbf{a}^-} \cdot W \\ &= \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} \left(x^{(\mathbf{a}+\mathbf{e})^+} \partial^{(\mathbf{a}+\mathbf{e})^-} \left(\prod_{\mathbf{a}_i < 0} (\theta_i + \mathbf{a}_i + 1) \right) \cdot W \right), \end{aligned}$$

and we deduce $K = \bigoplus_{\mathbf{a}\in\mathbb{Z}^d} x^{\mathbf{a}^+}\partial^{\mathbf{a}^-} \cdot K(\mathbf{a})$, where $K(\mathbf{a})$ is defined to be $L_0 + (\prod_{\mathbf{a}_i\le 0}(\theta_i + \mathbf{a}_i))\cdot W$. Thus, it is enough to consider the elements $f \in A$ of the form $x^{\mathbf{a}^+}\partial^{\mathbf{a}^-} g$ with $g \in W$ and prove that $g \in K(\mathbf{a})$. Moreover, we may assume that $m + \mathbf{a}_i > 0$ for $1 \le i \le d$.

By induction on r, we see that $x_i^r\partial_i^r = \prod_{j=1}^r(\theta_i - j + 1)$ for $r \ge 1$. Hence, for $\mathbf{a}_i < 0$, we have

$$x_i^m\partial_i^{-\mathbf{a}_i} = x_i^{m+\mathbf{a}_i}x_i^{-\mathbf{a}_i}\partial_i^{-\mathbf{a}_i} = x_i^{m+\mathbf{a}_i}\cdot\prod_{j=1}^{-\mathbf{a}_i}(\theta_i - j + 1)$$

and, for $\sigma \in \Delta$, we obtain

$$(x^{\hat{\sigma}})^m x^{\mathbf{a}^+}\partial^{\mathbf{a}^-} = \left(x^{(\mathbf{a}+m\hat{\sigma})^+}\partial^{(\mathbf{a}+m\hat{\sigma})^-}\right)\cdot\prod_{i\in\widehat{\Lambda}}\prod_{j=1}^{-\mathbf{a}_i}(\theta_i - j + 1),$$

where $\widehat{\Lambda} = \{i : \mathbf{v}_i \notin \sigma \text{ and } \mathbf{a}_i < 0\}$. We deduce

$$\left(\prod_{i\in\widehat{\Lambda}}\prod_{j=1}^{-\mathbf{a}_i}(\theta_i - j + 1)\right)\cdot g \in L_0 + \left(\prod_{i\in\Lambda}(\theta_i + \mathbf{a}_i)\right)\cdot W, \tag{10}$$

where $\Lambda = \{i : \mathbf{v}_i \in \sigma \text{ and } \mathbf{a}_i \le 0\}$.

For each $\mathbf{b} \in \mathbb{Z}^d$, we define an automorphism $\alpha_{\mathbf{b}} : W \longrightarrow W$ given by $\alpha_{\mathbf{b}}(\theta_i) = \theta_i - \mathbf{b}_i$ and we define $\rho_{\mathbf{b}} : W \longrightarrow W'$ to be the composition $\rho_{\mathbf{b}} = \rho\circ\alpha_{\mathbf{b}}$ (the map ρ is defined in the paragraph before Lemma 3.5). It is clear that $\rho_{\mathbf{b}}$ is surjective and $\mathrm{Ker}(\rho_{\mathbf{b}}) = L_0$. Applying $\rho_{\mathbf{b}}$ to Eq. (10) gives

$$\left(\prod_{i\in\widehat{\Lambda}}\prod_{j=1}^{-\mathbf{a}_i}\rho_{\mathbf{b}}(\theta_i - j + 1)\right)\cdot\rho_{\mathbf{b}}(g) \in \left(\prod_{i\in\Lambda}\rho_{\mathbf{b}}(\theta_i + \mathbf{a}_i)\right)\cdot W'.$$

Lemma 3.6 obviously extends to $\rho_{\mathbf{b}}$ and implies $\rho_{\mathbf{b}}(g) \in (\prod_{i\in\Lambda}\rho_{\mathbf{b}}(\theta_i + \mathbf{a}_i))\cdot W'$. Since this relation holds for every $\sigma \in \Delta$, a second application of Lemma 3.6 shows that $\rho_{\mathbf{b}}(g) \in (\prod_{\mathbf{a}_i\le 0}\rho_{\mathbf{b}}(\theta_i + \mathbf{a}_i))\cdot W'$ and therefore $g \in K(\mathbf{a}) = L_0 + (\prod_{a_i\le 0}(\theta_i + \mathbf{a}_i))\cdot W$. ∎

4. $\mathscr{D}$-MODULES

We now use Theorem 3.4 to describe the relation between A-modules and $\mathscr{D}$-modules on X. We begin by showing that the S-module associated to a $\mathscr{D}$-module has a graded A-module structure.

PROPOSITION 4.1. *If $\mathscr{F}$ is a left $\mathscr{D}$-module, then the graded S-module*

$$\Gamma_L(\mathscr{F}) = \bigoplus_{\bar{\mathbf{a}}\in\mathrm{Cl}(X)} H^0(X, \mathscr{O}(\bar{\mathbf{a}})\otimes\mathscr{F})$$

has a graded left A-module structure, extending the left S-module structure. Similarly, if $\mathcal{G}$ is a right $\mathcal{D}$-module, then the graded S-module

$$\Gamma_R(\mathcal{G}) = \bigoplus_{\bar{\mathbf{b}} \in \mathrm{Cl}(X)} H^0(X, \mathcal{G} \otimes \mathcal{O}(\bar{\mathbf{b}}))$$

has a graded right A-module structure, extending the right S-module structure.

Proof. We present the left $\mathcal{D}$-modules case here: the proof for right $\mathcal{D}$-modules is completely analogous. For the first assertion, it is enough as well to construct k-linear maps

$$\mu^{\mathcal{F}}_{\bar{\mathbf{a}}', \bar{\mathbf{a}}} : A_{\bar{\mathbf{a}}'} \otimes_k H^0(X, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{F}) \longrightarrow H^0(X, \mathcal{O}(\bar{\mathbf{a}} + \bar{\mathbf{a}}') \otimes \mathcal{F}),$$

for all $\bar{\mathbf{a}}, \bar{\mathbf{a}}' \in \mathrm{Cl}(X)$, satisfying the obvious axioms. To accomplish this, we consider a local version of the left multiplication map (3) when $\bar{\mathbf{b}} = 0$. More explicitly, for each $\sigma \in \Delta$, this morphism is

$$\mu_{\bar{\mathbf{a}}', \bar{\mathbf{a}}}|_{U_\sigma} : A_{\bar{\mathbf{a}}'} \otimes_k H^0(U_\sigma, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D}) \longrightarrow H^0(U_\sigma, \mathcal{O}(\bar{\mathbf{a}} + \bar{\mathbf{a}}') \otimes \mathcal{D}),$$

given by $\mu_{\bar{\mathbf{a}}', \bar{\mathbf{a}}}|_{U_\sigma}(f \otimes s) = f \circ s$. We claim that $\mu_{\bar{\mathbf{a}}', \bar{\mathbf{a}}}|_{U_\sigma}$ is a morphism of right $H^0(U_\sigma, \mathcal{D})$-modules. To see this, recall that sections $g \in H^0(U_\sigma, \mathcal{D})$ and $s \in H^0(U_\sigma, \mathcal{O}(\bar{\mathbf{b}}) \otimes \mathcal{D})$ can be identified with elements of the modules $\mathrm{Hom}_k((S_{x^{\hat{\sigma}}})_0, (S_{x^{\hat{\sigma}}})_0)$ and $\mathrm{Hom}_k((S_{x^{\hat{\sigma}}})_0, (S_{x^{\hat{\sigma}}})_{\bar{\mathbf{b}}})$ respectively. In particular, we have $(\mu_{\bar{\mathbf{a}}', \bar{\mathbf{a}}}|_{U_\sigma}(f \otimes s)) \cdot g = f \circ s \circ g = \mu_{\bar{\mathbf{a}}', \bar{\mathbf{a}}}|_{U_\sigma}(f \otimes s \cdot g)$, for all $f \in A_{\bar{\mathbf{a}}'}$. It follows that, by tensoring the map $\mu_{\bar{\mathbf{a}}', \bar{\mathbf{a}}}|_{U_\sigma}$ on the right with $H^0(U_\sigma, \mathcal{F})$ over $H^0(U_\sigma, \mathcal{D})$, we obtain a k-linear map

$$\mu^{\mathcal{F}}_{\bar{\mathbf{a}}', \bar{\mathbf{a}}}|_{U_\sigma} : A_{\bar{\mathbf{a}}'} \otimes_k H^0(U_\sigma, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{F}) \longrightarrow H^0(U_\sigma, \mathcal{O}(\bar{\mathbf{a}}' + \bar{\mathbf{a}}) \otimes \mathcal{F}).$$

These maps glue together to give $\mu^{\mathcal{F}}_{\bar{\mathbf{a}}', \bar{\mathbf{a}}}$, which makes $\Gamma_L(\mathcal{F})$ into a graded left A-module. ∎

Let A-GrMod_θ be the category of graded left A-modules F such that

$$(\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle) \cdot F_{\bar{\mathbf{a}}} = 0 \qquad \text{for all } \bar{\mathbf{a}} \in \mathrm{Cl}(X) \text{ and all } \bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee. \tag{11}$$

A graded A-module F is called $\mathfrak{b}$-torsion if, for every $f \in F$, there exists $\ell > 0$ such that $\mathfrak{b}^\ell f = 0$. Let $\mathfrak{b}$-Tors denote the full subcategory of $\mathfrak{b}$-torsion modules. Similarly, GrMod_θ-A is the category of graded right A-modules G satisfying $G_{\bar{\mathbf{b}}} \cdot (\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle) = 0$ for all $\bar{\mathbf{b}} \in \mathrm{Cl}(X)$ and all $\bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee$. Let Tors-$\mathfrak{b}$ denote the full subcategory of $\mathfrak{b}$-torsion modules in GrMod_θ-A.

It is clear that A-GrMod_θ and GrMod_θ-A are both abelian categories closed under taking graded subquotients. The main result in this section is

THEOREM 4.2. *The map $F \mapsto \widetilde{F}$ is an exact functor from A-GrMod_θ to $\mathscr{D}$-Mod, $\mathscr{F} \mapsto \Gamma_L(\mathscr{F})$ is a left exact functor from D-Mod to A-GrMod_θ, and there are natural equivalences of the functors*

$$\mathrm{id}_{A\text{-}\mathsf{GrMod}_\theta/\mathfrak{b}\text{-}\mathsf{Tors}} \xrightarrow{\cong} \Gamma_L \circ {}^\sim \ \textit{and}\ {}^\sim \circ \Gamma_L \xrightarrow{\cong} \mathrm{id}_{\mathscr{D}\text{-}\mathsf{Mod}}.$$

Similarly, the map $G \mapsto \widetilde{G}$ is an exact functor from GrMod_θ-A to Mod-$\mathscr{D}$, $\mathscr{G} \mapsto \Gamma_R(\mathscr{G})$ is a left exact functor from Mod-$\mathscr{D}$ to GrMod_θ-A, and there are natural equivalences of the functors

$$\mathrm{id}_{\mathsf{GrMod}_\theta\text{-}A/\mathsf{Tors}\text{-}\mathfrak{b}} \xrightarrow{\cong} \Gamma_R \circ \widetilde{\ \ } \quad \textit{and} \quad \widetilde{\ \ } \circ \Gamma_R \xrightarrow{\cong} \mathrm{id}_{\mathsf{Mod}\text{-}\mathscr{D}}.$$

In particular, every left $\mathscr{D}$-module is of the form $\widetilde{F}$ for some graded left A-module F and every right $\mathscr{D}$-module is of the form $\widetilde{G}$ for some graded right A-module G.

Proof of Theorem 4.2. Again, we give the proof only for left modules. For the first part, we consider an object F in A-GrMod_θ. By definition, we have $H^0(U_\sigma, \widetilde{F}) = (F_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$, where $(F_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$ is a left $(A_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$-module and $\sigma \in \Delta$. In light of Theorem 3.4, we must show that, for every $\bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee$, we have $\theta_{\bar{\mathbf{u}}} \cdot (F_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}} = 0$. Now, if $f/(x^{\hat{\sigma}})^m \in (F_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$, then $f \in F_{m\bar{\mathbf{e}}_\sigma}$ where $\mathbf{e}_\sigma = \sum_{i\in\hat{\sigma}} \mathbf{e}_i$ and the hypotheses on F imply

$$\begin{aligned}
\theta_{\bar{\mathbf{u}}} \cdot \frac{f}{(x^{\hat{\sigma}})^m} &= \left(\theta_{\bar{\mathbf{u}}} \frac{1}{(x^{\hat{\sigma}})^m}\right) \cdot f \\
&= \frac{1}{(x^{\hat{\sigma}})^m} \theta_{\bar{\mathbf{u}}} \cdot f - \sum_{i\in\hat{\sigma}} m\langle \bar{\mathbf{u}}, \bar{\mathbf{e}}_i\rangle \frac{1}{(x^{\hat{\sigma}})^m} \cdot f \\
&= \frac{1}{(x^{\hat{\sigma}})^m} (\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, m\bar{\mathbf{e}}_\sigma\rangle) \cdot f = 0.
\end{aligned}$$

Therefore $H^0(U_\sigma, \widetilde{F})$ has a structure of a left module over $H^0(U_\sigma, \mathscr{D})$ for every $\sigma \in \Delta$. It is straightforward to verify that these structures glue together to give a $\mathscr{D}$-module structure on $\widetilde{F}$. By construction, we see that $\widetilde{F}$ is a quasi-coherent sheaf over $\mathscr{D}$.

Conversely, let $\mathscr{F}$ be an object of $\mathscr{D}$-Mod. Applying Proposition 4.1, we know that $F = \Gamma_L(\mathscr{F})$ is graded left A-module, so it is enough to prove that F satisfies (11). Fixing $\bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee$ and $\bar{\mathbf{a}} \in \mathrm{Cl}(X)$, it suffices to show that $\mu^{\mathscr{F}}_{\bar{\mathbf{0}},\bar{\mathbf{a}}}|_{U_\sigma}((\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}}\rangle) \otimes s') = 0$, for every $\sigma \in \Delta$ and all sections $s' \in H^0(U_\sigma, \mathscr{O}(\bar{\mathbf{a}}) \otimes \mathscr{F})$. As explained in Proposition 4.1, we have

$$\begin{aligned}
H^0(U_\sigma, \mathscr{O}(\bar{\mathbf{a}}) \otimes \mathscr{F}) &= H^0(U_\sigma, \mathscr{O}(\bar{\mathbf{a}}) \otimes \mathscr{D} \otimes_{\mathscr{D}} \mathscr{F}) \\
&= H^0(U_\sigma, \mathscr{O}(\bar{\mathbf{a}}) \otimes \mathscr{D}) \otimes_{H^0(U_\sigma, \mathscr{D})} H^0(U_\sigma, \mathscr{F}),
\end{aligned}$$

so that we may identify s' with a linear combination of elements of the form $s \otimes f$. By definition, we have

$$\mu^{\mathscr{F}}_{\bar{\mathbf{0}},\,\bar{\mathbf{a}}}|_{U_\sigma}((\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle) \otimes s \otimes f) = (\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle) \circ s \otimes f,$$

and we claim that $(\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle) \circ s$ is zero. Indeed, for every $x^{\mathbf{c}} \in (S_{x^{\hat{\sigma}}})_{\bar{\mathbf{a}}}$ we have $\theta_{\bar{\mathbf{u}}}(x^{\mathbf{c}}) = \langle \bar{\mathbf{u}}, \bar{\mathbf{e}}_1 \rangle c_1 x^{\mathbf{c}} + \cdots + \langle \bar{\mathbf{u}}, \bar{\mathbf{e}}_d \rangle c_d x^{\mathbf{c}} = \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle x^{\mathbf{c}}$. Therefore, if $g \in H^0(U_\sigma, \mathscr{O}) = (S_{x^{\hat{\sigma}}})_0$, then $s(g) \in (S_{x^{\hat{\sigma}}})_{\bar{\mathbf{a}}}$ and

$$(\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle)(s(g)) = \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle s(g) - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle s(g) = 0.$$

Finally, the exact sequence (8) provides the first natural transformation, once we observe that $H^i_{\mathfrak{b}}(F)$ is $\mathfrak{b}$-torsion. It follows from Cox [2] that the sheaf associated to $\Gamma_L(\mathscr{F})$ is isomorphic to $\mathscr{F}$. ∎

As a corollary, we obtain

Proof of Theorem 1.1. This follows immediately from Theorem 4.2. ∎

We next turn our attention to coherent $\mathscr{D}$-modules and finitely generated graded A-modules. We write A-GrMod^f_θ and GrMod^f_θ-A for the full subcategories of A-GrMod_θ and GrMod_θ-A consisting of finitely generated A-modules.

PROPOSITION 4.3. *If F is an object in A-GrMod^f_θ, then $\widetilde{F}$ is a coherent left $\mathscr{D}$-module. Moreover, every coherent left $\mathscr{D}$-module is of the form $\widetilde{F}$ for some $F \in A$-GrMod^f_θ. Similarly, $G \in \mathsf{GrMod}^f_\theta$-$A$ implies that $\widetilde{G}$ belongs to Coh-$\mathscr{D}$ and every coherent right $\mathscr{D}$-module is isomorphic to $\widetilde{G}$ for some $G \in \mathsf{GrMod}^f_\theta$-$A$.*

Our proof is analogous to the $\mathscr{O}$-module case found in Cox [2].

Proof. Once again, we present the proof only for left modules. Suppose that F belongs to A-GrMod^f_θ. To establish that $\widetilde{F} \in \mathscr{D}$-$\mathsf{Coh}$, we have to check that, for every $\sigma \in \Delta$, $(F_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$ is finitely generated over $(A_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$. Thus, it suffices to show that, for every element $f \in F$, there is an invertible element $g \in S_{x^{\hat{\sigma}}}$ such that $g \cdot f$ has degree zero. Consider f in $F_{\bar{\mathbf{a}}}$. Since X is smooth, there is a divisor corresponding to $\mathbf{a} \in \mathbb{Z}^d$ supported outside σ and whose class is $\bar{\mathbf{a}}$. It follows that $x^{-\mathbf{a}}$ is an invertible element in $S_{x^{\hat{\sigma}}}$ and $x^{-\mathbf{a}} \cdot f$ has degree zero.

For the second assertion, we must show that given a coherent left $\mathscr{D}$-module $\mathscr{F}$ there exists a finitely generated A-submodule F of $\Gamma_L(\mathscr{F})$ such that $\widetilde{F} \cong \mathscr{F}$. Since $\mathscr{F}$ is coherent, $H^0(U_\sigma, \mathscr{F})$ is finitely generated over $H^0(U_\sigma, \mathscr{D})$, for every $\sigma \in \Delta$. Choose, for each $\sigma \in \Delta$, a finite set of homogeneous elements in $\Gamma_L(\mathscr{F})$ which are the numerators for a corresponding set of generators of $H^0(U_\sigma, \mathscr{F})$. Setting F to be the A-submodule of $\Gamma_L(\mathscr{F})$ generated by the union of these sets, we have $\widetilde{F} \cong \mathscr{F}$ and F is finitely generated over A. ∎

Remark 4.4. As a consequence of Theorem 4.2, we have $\widetilde{F} = 0$ if and only if the graded S-module F satisfies $F = H^0_{\mathfrak{b}}(F)$; this is equivalent to saying that F is a $\mathfrak{b}$-torsion module. At the other extreme, F has no $\mathfrak{b}$-torsion when $H^0_{\mathfrak{b}}(F) = 0$ and we say that F is $\mathfrak{b}$-saturated when $H^0_{\mathfrak{b}}(F) = H^1_{\mathfrak{b}}(F) = 0$. Now, every left $\mathcal{D}$-module $\mathcal{F}$ can be represented by a unique saturated A-module, namely $\Gamma_L(\mathcal{F})$. Unfortunately, this may not be finitely generated, even if $\mathcal{F}$ is coherent. However, by replacing F with a suitable submodule of $F/H^0_{\mathfrak{b}}(F)$, we may assume that F has no $\mathfrak{b}$-torsion and is finitely generated, whenever $\mathcal{F}$ is coherent.

EXAMPLE 4.5. The A-module corresponding to the structure sheaf $\mathcal{O}$ (which is a left $\mathcal{D}$-module) is $\Gamma_L(\mathcal{O}) = (A/A \cdot (\partial_1, \ldots, \partial_d))$. In particular, $\Gamma_L(\mathcal{O})$ is isomorphic to S, where S has the standard A-module structure.

COROLLARY 4.6. *For every* $\bar{\mathbf{b}} \in \mathrm{Cl}(X)$, *there is an isomorphism of graded left A-modules* $\Gamma_L(\mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}})) \cong D_L(\bar{\mathbf{b}})$. *Similarly, for every* $\bar{\mathbf{a}} \in \mathrm{Cl}(X)$, *we have* $\Gamma_R(\mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D}) \cong D_R(\bar{\mathbf{a}})$.

Proof. The proof of Theorem 3.4 provides

$$\eta_{(*,\bar{\mathbf{0}})} : D_L(\bar{\mathbf{b}}) \xrightarrow{\cong} \Gamma_L(\mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}}))$$

and

$$\eta_{(\bar{\mathbf{0}},*)} : D_R(\bar{\mathbf{a}}) \xrightarrow{\cong} \Gamma_R(\mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D}). \qquad \blacksquare$$

COROLLARY 4.7. *For* $\bar{\mathbf{a}}, \bar{\mathbf{b}} \in \mathrm{Cl}(X)$, *we have* $D_L(\bar{\mathbf{a}}) \in A\text{-}\mathsf{GrMod}_\theta$ *and* $D_R(\bar{\mathbf{b}}) \in \mathsf{GrMod}_\theta\text{-}A$.

Proof. This follows from Corollary 4.6 and Theorem 4.2. ∎

COROLLARY 4.8. *Let F be a graded left A-module generated by homogeneous elements* $\{f_i\}_i$ *with* $\deg f_i = \bar{\mathbf{b}}_i$. *For F to belong to* $A\text{-}\mathsf{GrMod}_\theta$, *it is necessary and sufficient that* $(\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{b}}_i \rangle) \cdot f_i = 0$ *for all i and all* $\bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee$. *A similar assertion holds for graded right A-modules.*

Proof. This condition is clearly necessary. To see the other direction, consider the surjective graded morphism defined by the given generators, $\bigoplus_i A(-\bar{\mathbf{b}}_i) \longrightarrow F$. By hypothesis, this factors to an epimorphism $\bigoplus_i D_L(-\bar{\mathbf{b}}_i) \longrightarrow F$ and Corollary 4.7 implies that $F \in A\text{-}\mathsf{GrMod}_\theta$. ∎

COROLLARY 4.9. *For each* $\mathcal{F} \in \mathcal{D}\text{-}\mathsf{Mod}$, *there exist a family* $\{\bar{\mathbf{b}}_i\}_i$ *of elements in* $\mathrm{Cl}(X)$ *and an epimorphism* $\bigoplus_i \mathcal{D} \otimes \mathcal{O}(-\bar{\mathbf{b}}_i) \longrightarrow \mathcal{F}$. *The analogous result also holds for* $\mathcal{G} \in \mathsf{Mod}\text{-}\mathcal{D}$.

Proof. Theorem 4.2 implies that there exists an $F \in A\text{-}\mathsf{GrMod}_\theta$ such that $\mathcal{F} \cong \widetilde{F}$. Now, Corollary 4.8 gives an epimorphism $\bigoplus_i D_L(-\bar{\mathbf{b}}_i) \longrightarrow \mathcal{F}$. Taking the corresponding morphism of sheaves and applying Corollary 4.6 establishes the claim. ∎

Remark 4.10. Applying Corollary 4.9, one can construct a resolution of a $\mathscr{D}$-module using the twisted modules $\mathscr{D} \otimes \mathscr{O}(\bar{\mathbf{b}}_i)$. More concretely, given a graded module F satisfying $\widetilde{F} \cong \mathscr{F}$, Gröbner basis techniques can be used to construct a resolution of F by modules $D_L(\bar{\mathbf{b}})$ which will then lift to a resolution of $\mathscr{F}$. It would be interesting to investigate the relationship between these two types of resolutions.

The last part of this section is devoted to the categorical equivalence between right and left $\mathscr{D}$-modules on the toric variety X. Recall that, for a smooth variety X of dimension n, the sheaf of differential forms of top degree Ω^n has a natural structure of a right $\mathscr{D}$-module extending the usual $\mathscr{O}$-module structure. Locally, right multiplication of an n-form ω with a vector field ν is defined by $\omega \cdot \nu = -\mathrm{Lie}_\nu(\omega)$, where $\mathrm{Lie}_\nu(\omega)$ is the Lie derivative of ω along ν; see Section VI.3.3 in Borel [1].

The equivalence of categories $\tau_{LR} : \mathscr{D}\text{-}\mathsf{Mod} \longrightarrow \mathsf{Mod}\text{-}\mathscr{D}$ and its inverse $\tau_{RL} : \mathsf{Mod}\text{-}\mathscr{D} \longrightarrow \mathscr{D}\text{-}\mathsf{Mod}$ are defined as follows: For a left $\mathscr{D}$-module $\mathscr{F}$, we have $\tau_{LR}(\mathscr{F}) = \mathscr{F} \otimes \Omega^n$ where the right multiplication with a vector field ν is $(f \otimes \omega) \cdot \nu = -\nu(f) \otimes \omega + f \otimes \omega \cdot \nu$. Similarly, if $\mathscr{G}$ is a right $\mathscr{D}$-module, then $\tau_{RL}(\mathscr{G}) = \mathscr{H}om_{\mathscr{O}}(\Omega^n, \mathscr{G})$ and left multiplication with a vector field ν is given by $\nu \cdot \psi(\omega) = \psi(\omega \cdot \nu) - \psi(\omega) \cdot \nu$.

In particular, the left–right equivalence of A-modules is given by the algebra involution $\tau : A \longrightarrow A$, where $x^{\mathbf{a}}\partial^{\mathbf{b}} \mapsto (-\partial)^{\mathbf{b}} x^{\mathbf{a}}$. Specifically, given a graded left A-module F, we obtain a graded right A-module F^τ which has the same underlying additive structure and has multiplication defined by $g \cdot f = \tau(f) \cdot g$ for $g \in A$ and $f \in F$. Similarly, if G is a graded right A-module, then an analogous procedure yields the left A-module G^τ. It is clear that $(F^\tau)^\tau = F$ and $(G^\tau)^\tau = G$. Furthermore, for graded A-modules, we have

PROPOSITION 4.11. *There are inverse equivalences of categories*

$$\tau_{LR}^{\mathsf{mod}} : A\text{-}\mathsf{GrMod}_\theta \longrightarrow \mathsf{GrMod}_\theta\text{-}A, \qquad \tau_{RL}^{\mathsf{mod}} : \mathsf{GrMod}_\theta\text{-}A \longrightarrow A\text{-}\mathsf{GrMod}_\theta$$

given by $\tau_{LR}^{\mathsf{mod}}(F) = F^\tau(-\bar{\mathbf{e}})$ *and* $\tau_{RL}^{\mathsf{mod}}(G) = G^\tau(\bar{\mathbf{e}})$ *where* $\bar{\mathbf{e}} \in \mathrm{Cl}(X)$ *is the class of* $\mathbf{e} = \mathbf{e}_1 + \cdots + \mathbf{e}_d \in \mathbb{Z}^d$.

Proof. We only need to show that the graded components of $\tau_{LR}^{\mathsf{mod}}(F)$ and $\tau_{RL}^{\mathsf{mod}}(G)$ are annihilated by suitable Euler operators. However, this follows from the fact that $\tau(\theta_{\bar{\mathbf{u}}}) = -\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{e}} \rangle$, where $\bar{\mathbf{u}} \in \mathrm{Cl}(X)$. ∎

EXAMPLE 4.12. Since $\tau(\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle) = -(\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} - \bar{\mathbf{e}} \rangle)$, there is an isomorphism $\tau_{LR}^{\mathsf{mod}}(D_L(\bar{\mathbf{a}})) \cong D_R(\bar{\mathbf{a}} - \bar{\mathbf{e}})$.

We next show that these equivalences of categories are compatible with the functors in Theorem 4.2. We will use the fact that, for a smooth toric variety X, there is a natural isomorphism $\Omega^n \cong \mathscr{O}(-\bar{\mathbf{e}})$. In fact, if 0 denotes the unique zero-dimensional cone in Δ, then this isomorphism identifies the section $dy_1/y_1 \wedge \cdots \wedge dy_n/y_n$ with $1/x_1 \cdots x_d$ on the open subset U_0; see Section 4.3 in Fulton [4].

PROPOSITION 4.13. *For the pair of functors* $(\tau_{LR}^{\mathrm{mod}}, \tau_{LR})$, *the diagrams*

$$
\begin{array}{ccc}
A\text{-}\mathsf{GrMod}_\theta & \xrightarrow{\tau_{LR}^{\mathrm{mod}}} & \mathsf{GrMod}_\theta\text{-}A \\
\downarrow \sim & & \downarrow \sim \\
\mathscr{D}\text{-}\mathsf{Mod} & \xrightarrow{\tau_{LR}} & \mathsf{Mod}\text{-}\mathscr{D}
\end{array}
\qquad
\begin{array}{ccc}
\mathscr{D}\text{-}\mathsf{Mod} & \xrightarrow{\tau_{LR}} & \mathsf{Mod}\text{-}\mathscr{D} \\
\downarrow \Gamma_L & & \downarrow \Gamma_R \\
A\text{-}\mathsf{GrMod}_\theta & \xrightarrow{\tau_{LR}^{\mathrm{mod}}} & \mathsf{GrMod}_\theta\text{-}A
\end{array}
$$

are commutative, up to natural isomorphisms. A similar statement holds for the pair $(\tau_{RL}^{\mathrm{mod}}, \tau_{RL})$.

Proof. Since $\tau_{RL} = (\tau_{LR})^{-1}$ and $\tau_{RL}^{\mathrm{mod}} = (\tau_{LR}^{\mathrm{mod}})^{-1}$, the second assertion is a consequence of the first. Because $F^\tau \cong F$ as S-modules and $\Omega^n \cong \mathscr{O}(-\bar{\mathbf{e}})$ as $\mathscr{O}$-modules, there is a natural isomorphism of $\mathscr{O}$-modules $\beta_F : \tau_{LR}(\widetilde{F}) = \widetilde{F} \otimes \Omega^n \longrightarrow \widetilde{\tau_{LR}^{\mathrm{mod}}(F)} = \widetilde{F^\tau(-\bar{\mathbf{e}})}$. Thus, it suffices to prove that β_F is compatible with the right $\mathscr{D}$-module structures.

By taking a presentation $\bigoplus_i D_L(\bar{\mathbf{b}}_i) \longrightarrow F$, we see that it suffices to establish the claim for $F = D_L(\bar{\mathbf{b}}_i)$. In this case, the restriction map

$$H^0(U, \widetilde{F^\tau(-\bar{\mathbf{e}})}) \longrightarrow H^0(U', \widetilde{F^\tau(-\bar{\mathbf{e}})})$$

is injective for open subsets $U' \subseteq U \subseteq X$. Thus, the claim reduces to showing that β_F is compatible with the right $\mathscr{D}$-module structure on U_0. Over U_0, the map $\beta_F|_{U_0} : (F_{x_1\cdots x_d})_{\bar{\mathbf{0}}} \otimes_{(S_{x_1\cdots x_d})_{\bar{\mathbf{0}}}} H^0(U_0, \Omega^n) \longrightarrow (F_{x_1\cdots x_d})^\tau_{-\bar{\mathbf{e}}}$ is given by $f \otimes \omega \mapsto f/x_1 \cdots x_d$, where $\omega = dy_1/y_1 \wedge \cdots \wedge dy_n/y_n$. Now, it is enough to check that $\beta_F|_{U_0}$ is compatible with right multiplication with a vector field ν over U_0. Using the notation from Lemma 3.5, we may assume that $\nu = y^{\mathbf{p}}\rho(\theta_i)$, for some $\mathbf{p} \in \mathbb{Z}^n$. We first compute

$$(f \otimes \omega) \cdot y^{\mathbf{p}}\rho(\theta_i) = -(y^{\mathbf{p}}\rho(\theta_i) \cdot f) \otimes \omega + f \otimes (\omega \cdot y^{\mathbf{p}}\rho(\theta_i)).$$

By definition, we have

$$
\begin{aligned}
(\omega \cdot y^{\mathbf{p}}\vartheta_j)(\eth_1, \ldots, \eth_n) &= -(\mathrm{Lie}_{y^{\mathbf{p}}\vartheta_j}(\omega))(\eth_1, \ldots, \eth_n) \\
&= \sum_{i=1}^{n} \omega(\eth_1, \ldots, [y^{\mathbf{p}}\vartheta_j, \eth_i], \ldots, \eth_n) \\
&\quad - \mathrm{Lie}_{y^{\mathbf{p}}\vartheta_j}(\omega(\eth_1, \ldots, \eth_n)) \\
&= -\sum_{i=1}^{n}\left(\delta_{ij}\frac{(\mathbf{p}_j + 1)y^{\mathbf{p}}}{y_1 \cdots y_n}\right) + \frac{y^{\mathbf{p}}}{y_1 \cdots y_n} \\
&= -\frac{\mathbf{p}_i y^{\mathbf{p}}}{y_1 \cdots y_n},
\end{aligned}
$$

from which we deduce that $\omega \cdot y^{\mathbf{p}}\rho(\theta_i) = -\iota(\mathbf{p})_i y^{\mathbf{p}} \cdot \omega$. Identifying the action of $y^{\mathbf{p}}\rho(\theta_i)$ on F with the action of $x^{\iota(\mathbf{p})}\theta_i$, we obtain

$$(f \otimes \omega) \cdot y^{\mathbf{p}}\rho(\theta_i) = \big((-x^{\iota(\mathbf{p})}\theta_i - \iota(\mathbf{p})x^{\iota(\mathbf{p})x^{\iota(\mathbf{p})}}) \cdot f\big) \otimes \omega.$$

On the other hand, we have

$$\begin{aligned} \frac{f}{x_1 \cdots x_d} \cdot x^{\iota(\mathbf{p})}\theta_i &= \tau(x^{\iota(\mathbf{p})}\theta_i) \cdot \frac{f}{x_1 \cdots x_d} \\ &= \big(-x^{\iota(\mathbf{p})}\theta_{\mathrm{i}} - (\iota(\mathbf{p})_i + 1)x^{\iota(\mathbf{p})}\big)\frac{f}{x_1 \cdots x_d} \\ &= \frac{1}{x_1 \cdots x_d}\big(-x^{\iota(\mathbf{p})}\theta_i - \iota(\mathbf{p})_i x^{\iota(\mathbf{p})}\big)f, \end{aligned}$$

and we conclude that $\beta_F((f \otimes \omega) \cdot \nu) = \beta_F(f \otimes \omega) \cdot \nu$.

In the second part, we consider $\mathscr{F} \in \mathscr{D}\text{-}\mathsf{Mod}$. For $F = \Gamma_L(\mathscr{F})$, we construct the natural map $\beta'_{\mathscr{F}} : \tau_{LR}^{\mathsf{mod}}(\Gamma_L(\mathscr{F})) \longrightarrow \Gamma_R(\tau_{LR}(\mathscr{F}))$, by composing the morphisms

$$\tau_{LR}^{\mathsf{mod}}(\Gamma_L(\mathscr{F})) = F^{\tau}(-\bar{\mathbf{e}}) \longrightarrow \Gamma_R(\widetilde{F^{\tau}(-\bar{\mathbf{e}})}), \tag{12}$$

$$\beta_{F^{\tau}(-\bar{\mathbf{e}})} : \Gamma_R(\widetilde{F^{\tau}(-\bar{\mathbf{e}})}) \longrightarrow \Gamma_R(\mathscr{F} \otimes \Omega^n) = \Gamma_R(\tau_{LR}(\mathscr{F})). \tag{13}$$

Since F is $\mathfrak{b}$-saturated, it follows that $F^{\tau}(-\bar{\mathbf{e}})$ is also $\mathfrak{b}$-saturated and hence that (12) is an isomorphism. Moreover, Γ_R is a functor and $\beta_{F^{\tau}(-\bar{\mathbf{e}})}$ is an isomorphism, which implies that (13) is also an isomorphism. ∎

EXAMPLE 4.14. The isomorphism of $\mathscr{O}$-modules $\Omega^n \cong \mathscr{O}(-\bar{\mathbf{e}})$ yields an isomorphism of S-modules $\Gamma_R(\Omega^n) \cong S(-\bar{\mathbf{e}})$. Proposition 4.13 shows that this is an isomorphism of right A-modules if $S(-\bar{\mathbf{e}})$ has the right A-module structure given by

$$S(-\bar{\mathbf{e}}) \cong \frac{A(-\bar{\mathbf{e}})}{(\partial_1, \ldots, \partial_d) \cdot A}.$$

5. THE CHARACTERISTIC VARIETY

In this section, we use the relationship between $\mathscr{D}$-modules on X and graded A-modules to describe the characteristic varieties. In particular, we relate the dimensions of F and $\widetilde{F}$. For simplicity, we restrict our attention to left modules.

We start by recalling the quotient construction of X; see Cox [2] or Musson [11]. Let T be the torus $\mathrm{Hom}(\mathrm{Cl}(X), k^*) \cong (k^*)^{d-n}$. The group T can be embedded into $(k^*)^d$ by the projection $\mathbb{Z}^d \longrightarrow \mathrm{Cl}(X)$. The diagonal action of $(k^*)^d$ on the affine space $\mathbb{A}^d$ induces an action of T on $\mathbb{A}^d$

such that the open subset $U = \mathbb{A}^d \backslash \mathrm{Var}(\mathfrak{b})$ is T-invariant; $\mathrm{Var}(\mathfrak{b})$ denotes the subscheme associated to the ideal $\mathfrak{b}$. Since X is smooth (and hence simplicial), there is a canonical morphism $U \longrightarrow X$ such that X is a geometric quotient of U with respect to the action of T. Furthermore, we have

LEMMA 5.1. *For every $z \in U$, $\mathrm{Stab}_T(z) = \{1\}$. In particular, all the T-orbits in U have dimension $d - n$.*

Proof. Consider a point z in U and $t \in T$ satisfying $t \cdot u = u$. Writing $z = (z_1, \ldots, z_d)$, we have $t \cdot z = (t(\bar{\mathbf{e}}_1)z_1, \ldots, t(\bar{\mathbf{e}}_d)z_d)$ and we deduce that $t(\bar{\mathbf{e}}_i) = 1$, for all i such that $z_i \neq 0$. Because there is a $\sigma \in \Delta$ such that $x^{\hat{\sigma}}(z) \neq 0$, we conclude that $t(\bar{\mathbf{e}}_i) = 1$ for every i with $\mathbf{v}_i \notin \sigma$.

On the other hand, t belongs to $\mathrm{Hom}(\mathrm{Cl}(X), k^*)$, so we have

$$t(\langle \mathbf{h}, \mathbf{v}_1 \rangle \bar{\mathbf{e}}_1 + \cdots + \langle \mathbf{h}, \mathbf{v}_d \rangle \bar{\mathbf{e}}_d) = t(\bar{\mathbf{e}}_1)^{\langle \mathbf{h}, \mathbf{v}_1 \rangle} \cdots t(\bar{\mathbf{e}}_d)^{\langle \mathbf{h}, \mathbf{v}_d \rangle} = 1,$$

for every $\mathbf{h} \in N^\vee$. It follows that $\Pi_{\mathbf{v}_i \in \sigma} t(\bar{\mathbf{e}}_i)^{\langle \mathbf{h}, \mathbf{v}_i \rangle} = 1$. Because this holds for each $\mathbf{h} \in N^\vee$ and the $\mathbf{v}_i$ form part of a basis of N, we also conclude that $t(\bar{\mathbf{e}}_i) = 1$ when $v_i \in \sigma$ and therefore $t = 1$. ∎

We next identify $\mathbb{A}^d \times \mathbb{A}^d$ with the cotangent bundle of $\mathbb{A}^d$ and consider the natural T-action on it. Let $S' = k[x_1, \ldots, x_d, \xi_1, \ldots, \xi_d]$, with the $\mathrm{Cl}(X)$-grading given by $\deg(x_i) = -\deg(\xi_i) = \bar{\mathbf{e}}_i$, be the coordinate ring of $\mathbb{A}^d \times \mathbb{A}^d$. Since the action of T on $\mathbb{A}^d$ is linear, it follows that the action of T on $\mathbb{A}^d \times \mathbb{A}^d$ is given by $t \cdot (z_1, z_2) = (t \cdot z_1, t^{-1} \cdot z_2)$. It is clear that $V = U \times \mathbb{A}^d \subset \mathbb{A}^d \times \mathbb{A}^d$ is invariant under the action of T. We construct its quotient as follows.

PROPOSITION 5.2. *There is a morphism $\pi\colon V \longrightarrow X'$ such that X' is the geometric quotient of V by the action of T. In addition, for every $z \in V$, we have $\mathrm{Stab}_T(z) = \{1\}$, implying that all the T-orbits in V have dimension $d - n$.*

Proof. The first step is to construct the morphism $\pi\colon V \longrightarrow X'$ as a categorical quotient—this is a local problem. For every $\sigma \in \Delta$, let $V_\sigma \subseteq V$ be the open subset defined by the non-vanishing of $x^{\hat{\sigma}}$. In other words, we have $V_\sigma = (U \backslash \mathrm{Var}(x^{\hat{\sigma}})) \times \mathbb{A}^d \subseteq V$, which is clearly T-invariant. Thus, the categorical quotient is locally $V_\sigma = \mathrm{Spec}(S'[(x^{\hat{\sigma}})^{-1}]^T)$. Since $t \cdot x^{\mathbf{a}} \xi^{\mathbf{b}} = t(\bar{\mathbf{a}} - \bar{\mathbf{b}}) x^{\mathbf{a}} \xi^{\mathbf{b}}$, for every $t \in T$ and $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^d$, we have $S'[(x^{\hat{\sigma}})^{-1}]^T = S'[(x^{\hat{\sigma}})^{-1}]_{\bar{\mathbf{0}}}$. Now, if σ_0 is a face of σ such that $\sigma_0 = \sigma \cap \mathbf{h}^\perp$ for $\mathbf{h} \in N^\vee \cap \sigma^\vee$, then we set $\mathbf{c} = \langle \mathbf{h}, \mathbf{v}_1 \rangle \mathbf{e}_1 + \cdots + \langle \mathbf{h}, \mathbf{v}_d \rangle \mathbf{e}_d \in \mathbb{Z}^d$. It follows that $S'[(x^{\hat{\sigma}_0})^{-1}]_{\bar{\mathbf{0}}} = (S'[(x^{\hat{\sigma}})^{-1}]_{\bar{\mathbf{0}}})_{x^{\mathbf{c}}}$, which provides an open immersion $V_{\sigma_0} \hookrightarrow V_\sigma$. Thus, we obtain morphisms $\pi_\sigma\colon V_{\sigma_0} \longrightarrow V_\sigma$ which glue together to give the categorical quotient $\pi\colon V \longrightarrow X'$.

In the second step, we establish that $\pi\colon V \longrightarrow X'$ is in fact a geometric quotient. By Amplification 1.3 in Mumford, et al. [8], it suffices to show that every T-orbit in V is closed. Consider $z = (z_1, z_2) \in V$. Since $U \longrightarrow X$ is a geometric quotient, the projection $V \longrightarrow U$ induces a morphism $\chi\colon \overline{Tz} \longrightarrow \overline{Tz_1} = Tz_1$. By Lemma 5.1, the morphism $\gamma\colon T \longrightarrow Tz_1$ given by $\gamma(t) = tz_1$ is bijective. Because the characteristic of the ground field k is zero and both T and Tz_1 are smooth, the map γ is an isomorphism. Let $\gamma'\colon T \longrightarrow Tz \subseteq V$ be defined by $\gamma'(t) = tz$. Hence, the map $\gamma' \circ \gamma^{-1} \circ \chi\colon \overline{Tz} \longrightarrow Tz \subseteq \overline{Tz}$ is the identity on Tz. It follows that $\gamma' \circ \gamma^{-1} \circ \chi$ is the identity map on $\overline{Tz}$ and we conclude that $Tz = \overline{Tz}$.

Since the projection from V onto U is T-equivariant, the second assertion follows from Lemma 5.1. ∎

Before discussing characteristic varieties, we review some properties of the order filtration. Recall that the sheaf $\mathscr{D}$ is naturally filtered by the order of the differential operators. In particular, this makes $H^0(U_\sigma, \mathscr{D})$ into a filtered ring. We can also filter the ring A by the order of the differential operators (in fact, $S' = \operatorname{gr}(A)$), and this induces a filtration on the quotient

$$\frac{(A_{x^{\hat\sigma}})_{\bar{\mathbf{0}}}}{(A_{x^{\hat\sigma}})_{\bar{\mathbf{0}}} \cdot (\theta_{\bar{\mathbf{u}}} : \bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee)}.$$

As Musson observed, we have

LEMMA 5.3 (Musson). *For every $\sigma \in \Delta$, the isomorphism $\bar\varphi_\sigma$ (see Eq. (5)) preserves the filtrations induced by the order of differential operators.*

Proof. See Section 4 in Musson [10]. ∎

A filtration of an A-module F is called good if the associated graded module $\operatorname{gr}(F)$ is finitely generated over S'. Every finitely generated A-module has a good filtration and, conversely, any module with a good filtration is necessarily finitely generated over A. For a good filtration of F, we define the characteristic ideal $\mathfrak{i}(F)$ to be the radical of $\operatorname{Ann}_{S'}(\operatorname{gr}(F))$. Since any two good filtrations are equivalent, the characteristic ideal $\mathfrak{i}(F)$ is independent of the choice of a good filtration. The characteristic variety of F is $\operatorname{Ch}(F) = \operatorname{Var}(\mathfrak{i}(F)) \subseteq \mathbb{A}^d \times \mathbb{A}^d$. Analogously, for a $\mathscr{D}$-module $\mathscr{F}$ with good filtration, we define the characteristic variety $\operatorname{Ch}(\mathscr{F})$ to be the support of the associated graded sheaf $\operatorname{gr}(\mathscr{F})$.

We first describe the characteristic variety associated to the graded left A-modules $D_L(\bar{\mathbf{b}})$. Let $p_{\bar{\mathbf{u}}} = \langle \bar{\mathbf{u}}, \bar{\mathbf{e}}_1\rangle x_1\xi_1 + \cdots + \langle \bar{\mathbf{u}}, \bar{\mathbf{e}}_d\rangle x_d\xi_d \in S'$, for all $\bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee$. We consider the ideal $\mathfrak{p} = (p_{\bar{\mathbf{u}}} : \bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee)$ and the corresponding variety $Z = \operatorname{Var}(\mathfrak{p}) \subseteq \mathbb{A}^d \times \mathbb{A}^d$. It is clear that Z is invariant under the T-action.

PROPOSITION 5.4. *The variety Z is a normal complete intersection of dimension $d + n$. Moreover, Z equals the characteristic variety $\operatorname{Ch}(D_L(\bar{\mathbf{b}}))$, for every $\bar{\mathbf{b}} \in \mathrm{Cl}(X)$.*

Proof. By choosing a basis $\bar{\mathbf{u}}_1, \ldots, \bar{\mathbf{u}}_{d-n}$ for $\mathrm{Cl}(X)^\vee$, we can write

$$\mathfrak{p} = (p_{\bar{\mathbf{u}}_i} : 1 \le i \le d-n).$$

For each $\bar{\mathbf{u}}_i$, we pick a representative $\mathbf{u}_i \in (\mathbb{Z}^d)^\vee$ such that $\mathbf{u}_1, \ldots, \mathbf{u}_{d-n}$ are linearly independent. We then enlarge this collection to obtain a basis $\mathbf{u}_1, \ldots, \mathbf{u}_d$ for $(\mathbb{Z}^d)^\vee$. Setting $q_{\mathbf{u}} = \langle \mathbf{u}, \mathbf{e}_1\rangle x_1\xi_1 + \cdots + \langle \mathbf{u}, \mathbf{e}_d\rangle x_d\xi_d \in S'$, for all $\mathbf{u} \in (\mathbb{Z}^d)^\vee$, it follows that the ideal $(q_{\mathbf{u}_i} : 1 \le i \le d)$ equals $(x_i\xi_i : 1 \le i \le d)$, which has height d. Hence, the $q_{\mathbf{u}_i}$ and $p_{\mathbf{u}_i}$ form a regular sequence and we deduce $\dim(Z) = d + n$.

To prove that Z is normal, we apply Serre's criterion. Because being a complete intersection implies the (S2) condition, it suffices to show that Z satisfies condition (R1), which we check by using the Jacobian criterion. The Jacobian matrix $\mathrm{Jac}(x, \xi)$ of $(p_{\bar{\mathbf{u}}_1}, \ldots, p_{\bar{\mathbf{u}}_{d-n}})$ is given by

$$\begin{pmatrix} \langle \bar{\mathbf{u}}_1, \bar{\mathbf{e}}_1\rangle \xi_1 & \cdots & \langle \bar{\mathbf{u}}_1, \bar{\mathbf{e}}_d\rangle \xi_d & \langle \bar{\mathbf{u}}_1, \bar{\mathbf{e}}_1\rangle x_1 & \cdots & \langle \bar{\mathbf{u}}_1, \bar{\mathbf{e}}_d\rangle x_d \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \langle \bar{\mathbf{u}}_{d-n}, \bar{\mathbf{e}}_1\rangle \xi_1 & \cdots & \langle \bar{\mathbf{u}}_{d-n}, \bar{\mathbf{e}}_d\rangle \xi_d & \langle \bar{\mathbf{u}}_{d-n}, \bar{\mathbf{e}}_1\rangle x_1 & \cdots & \langle \bar{\mathbf{u}}_{d-n}, \bar{\mathbf{e}}_d\rangle x_d \end{pmatrix}.$$

Observe that, for $1 \le i \le d$, the restriction $\bigoplus_{j \ne i} \mathbb{Z}\mathbf{e}_j \longrightarrow \mathrm{Cl}(X)$ is surjective. Indeed, if σ_i is the cone generated by $\mathbf{v}_i$, then every element in $\mathrm{Cl}(X)$ can be represented by a divisor whose support does not intersect U_{σ_i}. We deduce that if the rank of $\mathrm{Jac}(x, \xi)$ is strictly less than $d - n$, then at least two of the pairs of coordinates $(x_1, \xi_1), \ldots, (x_d, \xi_d)$ are zero. By cutting with n extra quadrics $q_{\mathbf{u}_{d-n+1}}, \ldots, q_{\mathbf{u}_d}$, we see that the codimension of the singular locus of Z is at least two. Therefore, the variety Z is normal. Moreover, Z is a cone which implies that it is connected and hence integral.

Recall that, for $\bar{\mathbf{b}} \in \mathrm{Cl}(X)^\vee$, we have $D_L(\bar{\mathbf{b}}) = A(\bar{\mathbf{b}})/(\theta_{\bar{\mathbf{u}}_i} + \langle \bar{\mathbf{u}}_i, \bar{\mathbf{b}}\rangle : 1 \le i \le d-n)$, and for the order filtration the initial term (or principal symbol) of the above elements is $\mathrm{in}(\theta_{\bar{\mathbf{u}}_i} + \langle \bar{\mathbf{u}}_i, \bar{\mathbf{b}}\rangle) = \theta_{\bar{\mathbf{u}}_i}$. Since the $\theta_{\bar{\mathbf{u}}_i}$ for $1 \le i \le d-n$ form a regular sequence in $S' = \mathrm{gr}(A)$, it follows that $\mathrm{gr}(D_L(\bar{\mathbf{b}})) = S'/\mathfrak{p}$. On the other hand, we have already seen that $\mathfrak{p}$ is reduced, so we have $\mathrm{Ch}(D_L(\bar{\mathbf{b}})) = \mathrm{Var}(\mathfrak{p}) = Z$. ∎

COROLLARY 5.5. *If $F \in A\text{-}\mathsf{GrMod}^f_\theta$, then we have* $\mathrm{Ch}(F) \subseteq Z$.

Proof. By Corollary 4.8, F is a quotient of $\bigoplus_{i=1}^r D_L(\bar{\mathbf{b}}_i)$, for some r and some $\bar{\mathbf{b}}_i$. It follows that $\mathrm{Ch}(F) \subseteq \bigcup_{i=1}^r \mathrm{Ch}(D_L(\bar{\mathbf{b}}_i)) = Z$. ∎

We next relate the cotangent bundle of X to the variety Z. Consider the diagram

$$\begin{array}{ccccc} Z \cap V & \hookrightarrow & V & \longrightarrow & U \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi} & & \big\downarrow \\ \pi(Z \cap V) & \hookrightarrow & X' & \longrightarrow & X, \end{array}$$

where $X' \longrightarrow X$ arises from the universal property of the categorical quotient.

PROPOSITION 5.6. *There is a canonical isomorphism of varieties over X between $\zeta\colon \pi(Z \cap V) \longrightarrow X$ and the cotangent bundle T^*X over X.*

Proof. Since T^*X is naturally isomorphic to $\mathrm{Ch}(\mathscr{D})$, we see that T^*X is isomorphic to $\mathrm{Spec}\,(\mathrm{gr}H^0(U_\sigma, \mathscr{D}))$ over U_σ. On the other hand, from the local description of X', we know that the inverse image of U_σ is $\mathrm{Spec}(S'_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$ and therefore $\zeta^{-1}(U_\sigma) = \mathrm{Spec}((S'_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}/(p_{\bar{\mathbf{u}}} : \bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee))$. By Lemma 5.3, we have an isomorphism of filtered rings,

$$\bar{\varphi}_\sigma : \frac{(A_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}}{(A_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}} \cdot (\theta_{\bar{\mathbf{u}}} : \bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee)} \longrightarrow H^0(U_\sigma, \mathscr{D}).$$

Note that the graded ring associated to the left-hand side is $(S'_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}/(p_{\bar{\mathbf{u}}} : \bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee)$. Indeed, following the proof of Proposition 5.4, we see that the initial terms of $\theta_{\bar{\mathbf{u}}_1}, \dots, \theta_{\bar{\mathbf{u}}_{d-n}}$ are equal to $p_{\bar{\mathbf{u}}_1}, \dots, p_{\bar{\mathbf{u}}_{d-n}}$ and form a regular sequence in $(S'_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$. Therefore, by passing to the associated graded rings, $\bar{\varphi}_\sigma$ induces the required isomorphism. Because the $\bar{\varphi}_\sigma$ are compatible with restriction, these local isomorphisms glue together to give the required isomorphism. ∎

We now present the main result in this section.

THEOREM 5.7. *If $F \in A\text{-}\mathsf{GrMod}^f_\theta$, then the characteristic variety of F is T-invariant and $\pi(\mathrm{Ch}(F)\backslash \mathrm{Var}(\mathfrak{b}) \times \mathbb{A}^d) = \mathrm{Ch}(\widetilde{F})$.*

Proof. Since $F \in A\text{-}\mathsf{GrMod}^f_\theta$, we may choose a finite set $f_1, \dots, f_r$ of homogeneous generators for F. By using these homogeneous elements to define a good filtration of F, it follows that $\mathrm{gr}(F)$ is a graded finitely generated S'-module. Therefore, both $\mathfrak{j}(F) = \mathrm{Ann}_{S'}(\mathrm{gr}(F))$ and its radical $\mathfrak{i} = \sqrt{\mathfrak{j}(F)}$ are graded ideals of S'. Recall that for every $t \in T$, we have $t \cdot x^{\mathbf{a}}\xi^{\mathbf{b}} = t(\bar{\mathbf{a}} - \bar{\mathbf{b}})x^{\mathbf{a}}\xi^{\mathbf{b}}$. We deduce that every subscheme defined by a graded ideal is T-invariant; in particular, $\mathrm{Ch}(F) = \mathrm{Var}(\mathfrak{i})$ is T-invariant.

To prove the second assertion, we argue locally and use the identification in Proposition 5.6. Over the open subset U_σ, the ideal defining $\mathrm{Ch}(F)\backslash \mathrm{Var}(\mathfrak{b}) \times \mathbb{A}^d \subseteq \mathrm{Spec}(S'_{x^{\hat{\sigma}}})$ is $\mathfrak{j} \cdot S'_{x^{\hat{\sigma}}}$. On the other hand, the characteristic variety of $\widetilde{F}$ over U_σ can be computed as follows: If $H^0(U_\sigma, \widetilde{F}) = (F_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$ has the good filtration induced by the images of $f_1, \dots, f_r$, then we obtain $\mathrm{gr}(H^0(U_\sigma, \widetilde{F})) \cong (\mathrm{gr}(F)_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$. To see that the annihilator of $\mathrm{gr}(H^0(U_\sigma, \widetilde{F}))$ in $(S'_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$ is $(\mathfrak{j}_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$, it is enough to observe that $\mathrm{gr}(F)_{x^{\hat{\sigma}}}$ can be generated by elements of degree zero. Since $\sqrt{(\mathfrak{j}(F)_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}} = (\mathfrak{i}_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$, we may identify $\pi(\mathrm{Ch}(F) \cap V_\sigma)$ with $\mathrm{Ch}(\widetilde{F}|_{U_\sigma})$. Because these identifications are compatible with restriction, they glue together to give the required isomorphism. ∎

As a corollary, we obtain

Proof of Theorem 1.2. This follows immediately from Theorem 5.7. ∎

We end the paper by relating the dimension of the A-module F and its associated $\mathcal{D}$-module $\widetilde{F}$. By definition, the local dimension of a $\mathcal{D}$-module $\mathcal{F}$ at a point $p \in X$ is equal to the Krull dimension of the associated graded module of $\mathcal{F}_p$ with respect to a good filtration respecting the order filtration of $\mathcal{D}_p$. It is also equal to the dimension of the characteristic variety of $\mathcal{F}_p$. The dimension of F is by definition the maximum of the local dimensions and is equivalently the dimension of $\mathrm{Ch}(F)$.

By Theorem 5.7, the dimension of $\widetilde{F}$ is equal to the maximum of the local dimensions of F over the open set $\mathbb{A}^d \backslash \mathrm{Var}(\mathfrak{b})$ minus the dimension $d-n$ of the orbits under the group action. Before showing that when F has no $\mathfrak{b}$-torsion we can express this dimension in terms of $\dim(F)$, we collect two lemmas.

LEMMA 5.8. *If F is a finitely generated left A-module, f is an element of S, and F' is a finitely generated A-submodule of $F[f^{-1}]$, then* $\dim(F') \leq \dim(F)$.

Proof. This claim follows immediately from standard results about the Gelfand–Kirillov dimension; see Propositions 8.3.2(i) and 8.3.14(iii) in McConnell and Robson [9]. ∎

PROPOSITION 5.9. *Let $F \in A\text{-}\mathsf{GrMod}_\theta^f$ and recall that $\mathfrak{b}$ is the irrelevant ideal in S. If F has no $\mathfrak{b}$-torsion, then we have*

$$\dim(F) = \max\{\dim(F_p) : p \in \mathbb{A}^d \backslash \mathrm{Var}(\mathfrak{b})\}.$$

Proof. Suppose otherwise; then we have $\dim(F) > \dim(F_z)$ for all $z \in \mathbb{A}^d \backslash \mathrm{Var}(\mathfrak{b})$. Let F' be the maximum submodule of F of dimension strictly less than $\dim(F)$. In other words, F' is the submodule consisting of all $f \in F$ such that $A \cdot f$ has dimension strictly less than $\dim(F)$. Since F' is a submodule, there is a short exact sequence

$$0 \longrightarrow F' \longrightarrow F \longrightarrow \frac{F}{F'} \longrightarrow 0.$$

By construction, F/F' has no non-zero submodules of dimension strictly less than $\dim(F)$, and, hence, the irreducible components of $\mathrm{Ch}(F/F')$ have dimension at least $\dim(F)$; see Smith [14]. By hypothesis, the irreducible components of $\mathrm{Ch}(F)$ of dimension $\dim(F)$ are contained inside $\zeta^{-1}(\mathrm{Var}(\mathfrak{b}))$ where $\zeta\colon T^*X \longrightarrow X$. Since $\mathrm{Ch}(F) = \mathrm{Ch}(F') \cup \mathrm{Ch}(F/F')$, it follows that the $\mathrm{Ch}(F/F')$ is contained inside $\zeta^{-1}(\mathrm{Var}(\mathfrak{b}))$. Moreover, the support of an A-module equals the projection of its characteristic variety (see Granger and Maisonobe [5]), which implies that F/F' is supported on $\mathrm{Var}(\mathfrak{b})$. Taking the long exact sequence in local cohomology, we have

$$0 \longrightarrow H^0_{\mathfrak{b}}(F') \longrightarrow H^0_{\mathfrak{b}}(F) \longrightarrow H^0_{\mathfrak{b}}\left(\frac{F}{F'}\right) \longrightarrow H^1_{\mathfrak{b}}(F') \longrightarrow \cdots.$$

By assumption, F has no $\mathfrak{b}$-torsion, so we have $H^0_{\mathfrak{b}}(F') = H^0_{\mathfrak{b}}(F) = 0$. Because F/F' is supported on $\mathrm{Var}(\mathfrak{b})$, we have $H^0_{\mathfrak{b}}(F/F') = F/F'$. Choosing a set of generators $\mathfrak{b} = (s_1, \ldots, s_r)$, the long exact sequence induces

$$0 \longrightarrow \frac{F}{F'} \longrightarrow \frac{\bigoplus_{i=1}^r F'[s_i^{-1}]}{F'} \longrightarrow \cdots.$$

Hence, F/F' is a finitely generated A-subquotient of $\bigoplus_{i=1}^r F'[s_i^{-1}]$. Therefore, Lemma 5.8 implies that $\dim(F/F') \leq \dim(F') < \dim(F/F')$, which yields a contradiction. ∎

Finally, we have

THEOREM 5.10. *If $F \in A\text{-}\mathsf{GrMod}^f_\theta$ has no $\mathfrak{b}$-torsion, then we have*

$$\dim(\widetilde{F}) = \dim(F) - d + n.$$

Proof. Applying Proposition 5.9, we see that $\dim(F)$ is the maximum of the local dimensions of F over $\mathbb{A}^d \backslash \mathrm{Var}(\mathfrak{b})$. Hence, the claim follows from Proposition 5.2 and Theorem 5.7. ∎

A coherent $\mathscr{D}$-module $\mathscr{F}$ is holonomic if $\dim(\mathscr{F}) \leq \dim X$.

COROLLARY 5.11. *If $F \in A\text{-}\mathsf{GrMod}^f_\theta$ is holonomic, then $\widetilde{F}$ is holonomic. Furthermore, every holonomic $\mathscr{D}$-module is of the form $\widetilde{F}$ for some holonomic $F \in A\text{-}\mathsf{GrMod}^f_\theta$.*

Proof. This follows immediately from Theorem 5.10 and Remark 4.4. ∎

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